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Abstract

We consider a bilateral oligopoly version of the Shapley window model with large traders, represented as atoms, and small traders, represented by an atomless part. For this model, we show that, when atoms have Leontievan utility functions, any Cournot-Nash allocation is a Walras allocation and, consequently, it is Pareto optimal.

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1 Introduction

Gabszewicz and Michel (1997) introduced the so-called model of bilateral oligopoly, representing an exchange economy with two commodities where each trader is endowed with only one of them. Different strategic market games proposed in the line of research initiated by Shapley and Shubik to model different type of noncooperative strategic interaction (see Shubik (1973), Shapley (1976), Shapley and Shubik (1977), and, for a survey of

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this line of research, Giraud (2003)) have been formulated also in terms of a bilateral oligopoly framework.

The model of bilateral oligopoly was analyzed, in the case of a finite number of traders, by Bloch and Ghosal (1997), Bloch and Ferrer (2001), Dickson and Hartley (2008), Amir and Bloch (2009), among others.

In this paper, we consider the mixed version of this model introduced by Codognato et al. (2015) and further analyzed by Busetto et al. (2018b): a mixed exchange economy à la Shitovitz (1973) is studied, where large traders are represented as atoms and small traders are represented by an atomless part; noncooperative exchange is formalized as in the Shapley window model, a strategic market game with complete markets which was first proposed informally by Lloyd S. Shapley and further studied by Sahi and Yao (1989), Codognato and Ghosal (2000), Busetto et al. (2011), Busetto et al. (2018a), among others.

In this framework, Codognato et al. (2015) showed a theorem establishing that, under the assumptions that all traders' utility functions are continuous, strongly monotone, quasi-concave, and measurable, and atoms' utility functions are also differentiable, a necessary and sufficient condition for a Cournot-Nash allocation to be a Walras allocation is that all atoms demand a null amount of one of the two commodities.

Moreover, these authors showed, through some examples, that their result may not hold also when the conditions which guarantee the equivalence between the core and the set of Walras allocations in Shitovitz (1973), namely that atoms are of the same type, i.e., have the same endowments and preferences, are satisfied. With those examples, they rather showed that their equivalence result crucially depends on the assumptions introduced on atoms' preferences.

Here, we go deeper into the role played by these assumptions and we wonder which are the consequences when some of them are weakened. We do so studying what happens when atoms are characterized by a type of preferences classical in the economic literature: that expressed by Leontie-vian utility functions. This class of functions is indeed the most commonly used to represent commodities which are perfect complements. As is well known, this functional form was first extensively used in production theory within the input-output analysis developed by Leontief (1941) and was later extended, by analogy, to consumer theory. More recently, a complete characterization of Leontie-vian preferences has been provided by Ninjbat (2010) and Voorneveld (2014), among others.

The main result of the paper is a theorem showing that, when traders in

the atomless part have utility functions which satisfy the assumptions made in Codognato et al. (2015) whereas atoms have Leontievan utility functions, any Cournot-Nash allocation is a Walras allocation, and, consequently, is Pareto optimal.

Since Leontievan utility functions are neither strongly monotone nor differentiable, this theorem implies that those assumptions are not necessary for a Cournot-Nash allocation to be a Walras allocation, and consequently Pareto optimal.

Our main theorem also implies that, at a Cournot-Nash allocation, which is always a Walras allocation, all atoms demand a strictly positive amount of both commodities. As discussed in the detail in the paper, this outcome is due to perfect complementarity, which, in our framework, prevents atoms from substituting the two commodities; in contrast, substitutability between the two commodities is just what causes atoms to obtain corner assignments at a Cournot-Nash equilibrium in Codognato et al. (2015).

In this paper, we also study the relationship between the mixed bilateral oligopoly versions of the Shapley window model and of another prototypical strategic market game in the Shapley and Shubik line of research. that proposed by Amir et al. (1990). This model can be in turn interpreted as another generalization to a complete market context of the well-known strategic market game with commodity money proposed by Dubey and Shubik (1978).

In this regard, Codognato et al. (2015) already proved that, in their mixed bilateral oligopoly model, the set of Cournot-Nash allocations of the Shapley window model coincides with the set of the Cournot-Nash allocations of both the model of Dubey and Shubik (1978) and its generalization proposed by Amir et al. (1990). This result is crucially based on the assumption that the atoms' utility functions are strongly monotone and consequently it cannot be extended to our Leontievan framework.

We then provide a new proof that the Cournot-Nash allocations of those three models coincide also when atoms' preferences are of the Leontievan type, thereby showing that our main theorems extend to all of them.

The paper is organized as follows. In Section 2, we introduce the mathematical model. In section 3, we prove our main theorem. In Section 4, we discuss the model. In Section 5, we draw some conclusions and we sketch some further lines of research.

2 The mathematical model

We consider a pure exchange economy with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space (T, \mathcal{T}, μ) , where T is the set of traders, \mathcal{T} is the σ -algebra of all μ -measurable subsets of T , and μ is a real valued, non-negative, countably additive measure defined on \mathcal{T} . We assume that (T, \mathcal{T}, μ) is finite, i.e., $\mu(T) < \infty$. This implies that the measure space (T, \mathcal{T}, μ) contains at most countably many atoms. Let T_1 denote the set of atoms and T_0 the atomless part of T . We assume that $\mu(T_1) > 0$ and $\mu(T_0) > 0$.¹ A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for “each” trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. A coalition is a nonnull element of \mathcal{T} . The word “integrable” is to be understood in the sense of Lebesgue.

In the exchange economy, there are two different commodities. A commodity bundle is a point in R_+^2 . An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}: T \rightarrow R_+^2$. There is a fixed initial assignment \mathbf{w} , satisfying the following assumption.

Assumption 1. *There is a coalition S such that $\mathbf{w}^1(t) > 0$, $\mathbf{w}^2(t) = 0$, for each $t \in S$, $\mathbf{w}^1(t) = 0$, $\mathbf{w}^2(t) > 0$, for each $t \in S^c$. Moreover, $\text{card}(S \cap T_1) \geq 2$, whenever $\mu(S \cap T_0) = 0$, and $\text{card}(S^c \cap T_1) \geq 2$, whenever $\mu(S^c \cap T_0) = 0$.²*

An allocation is an assignment \mathbf{x} such that $\int_T \mathbf{x}(t) d\mu = \int_T \mathbf{w}(t) d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t: R_+^2 \rightarrow R$, satisfying the following assumptions.

Assumption 2. *$u_t: R_+^2 \rightarrow R$ is continuous, strongly monotone, and quasi-concave, for each $t \in T_0$, and $u_t(x^1, x^2) = \min\{a_{t1}x^1, a_{t2}x^2\}$, with $a_{t1} > 0$ and $a_{t2} > 0$, for each $t \in T_1$.*

Let \mathcal{B} denote the Borel σ -algebra of R_+^2 . Moreover, let $\mathcal{T} \otimes \mathcal{B}$ denote the σ -algebra generated by the sets $E \times F$ such that $E \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 3. *$u: T \times R_+^2 \rightarrow R$, given by $u(t, x) = u_t(x)$, for each $t \in T$ and for each $x \in R_+^2$, is $\mathcal{T} \otimes \mathcal{B}$ -measurable.*

A price vector is a nonnull vector $p \in R_+^2$. A Walras equilibrium is a pair (p, \mathbf{x}) , consisting of a price vector p and an allocation \mathbf{x} such that

¹The symbol 0 denotes the origin of R_+^2 as well as the real number zero: no confusion will result.

² $\text{card}(A)$ denotes the cardinality of a set A .

$p\mathbf{x}(t) = p\mathbf{w}(t)$ and $u_t(\mathbf{x}(t)) \geq u_t(y)$, for all $y \in \{x \in R_+^2 : px = p\mathbf{w}(t)\}$, for each $t \in T$. A Walras allocation is an allocation \mathbf{x}^* for which there exists a price vector p^* such that the pair (p^*, \mathbf{x}^*) is a Walras equilibrium.

Borrowing from Codognato et al. (2015) and Busetto et al. (2018b), we introduce now the two-commodity version of the Shapley window model. A strategy correspondence is a correspondence $\mathbf{B} : T \rightarrow \mathcal{P}(R_+^4)$ such that, for each $t \in T$, $\mathbf{B}(t) = \{(b_{ij}) \in R_+^4 : \sum_{j=1}^2 b_{ij} \leq \mathbf{w}^i(t), i = 1, 2\}$. With some abuse of notation, we denote by $b(t) \in \mathbf{B}(t)$ a strategy of trader t , where $b_{ij}(t)$, $i, j = 1, 2$, represents the amount of commodity i that trader t offers in exchange for commodity j . A strategy selection is an integrable function $\mathbf{b} : T \rightarrow R_+^4$, such that, for each $t \in T$, $\mathbf{b}(t) \in \mathbf{B}(t)$. Given a strategy selection \mathbf{b} , we call aggregate matrix the matrix $\bar{\mathbf{B}}$ such that $\bar{b}_{ij} = (\int_T \mathbf{b}_{ij}(t) d\mu)$, $i, j = 1, 2$. Moreover, we denote by $\mathbf{b} \setminus b(t)$ the strategy selection obtained from \mathbf{b} by replacing $\mathbf{b}(t)$ with $b(t) \in \mathbf{B}(t)$ and by $\bar{\mathbf{B}} \setminus b(t)$ the corresponding aggregate matrix.

Consider the following two further definitions (see Sahi and Yao (1989)).

Definition 1. A nonnegative square matrix D is said to be irreducible if, for every pair (i, j) , with $i \neq j$, there is a positive integer k such that $d_{ij}^{(k)} > 0$, where $d_{ij}^{(k)}$ denotes the ij -th entry of the k -th power D^k of D .

Definition 2. Given a strategy selection \mathbf{b} , a price vector p is said to be market clearing if

$$p \in R_{++}^2, \sum_{i=1}^2 p^i \bar{b}_{ij} = p^j (\sum_{i=1}^2 \bar{b}_{ji}), j = 1, 2. \quad (1)$$

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector p satisfying (1) if and only if $\bar{\mathbf{B}}$ is irreducible. Then, we denote by $p(\mathbf{b})$ a function which associates with each strategy selection \mathbf{b} the unique, up to a scalar multiple, price vector p satisfying (1), if $\bar{\mathbf{B}}$ is irreducible, and is equal to 0 otherwise.

Given a strategy selection \mathbf{b} and a price vector p , consider the assignment determined as follows:

$$\begin{aligned} \mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t) - \sum_{i=1}^2 \mathbf{b}_{ji}(t) + \sum_{i=1}^2 \mathbf{b}_{ij}(t) \frac{p^i}{p^j}, \text{ if } p \in R_{++}^2, \\ \mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t), \text{ otherwise,} \end{aligned}$$

$j = 1, 2$, for each $t \in T$.

Given a strategy selection \mathbf{b} and the function $p(\mathbf{b})$, traders' final holdings are determined according to this rule and consequently expressed by the assignment

$$\mathbf{x}(t) = \mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b})),$$

for each $t \in T$.³ It is straightforward to show that this assignment is an allocation satisfying the budget constraint $p(\mathbf{b})\mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b})) = p(\mathbf{b})\mathbf{w}(t)$, for each $t \in T$.

We are now able to define the notion of a Cournot-Nash equilibrium for this reformulation of the Shapley window model (see Codognato and Ghosal (2000) and Busetto et al. (2011)).

Definition 3. *A strategy selection $\hat{\mathbf{b}}$ such that $\hat{\mathbf{B}}$ is irreducible is a Cournot-Nash equilibrium if*

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, b(t), p(\hat{\mathbf{b}} \setminus b(t)))),$$

for each $b(t) \in \mathbf{B}(t)$ and for each $t \in T$.

A Cournot-Nash allocation is an allocation $\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$, where $\hat{\mathbf{b}}$ is a Cournot-Nash equilibrium.

3 Cournot-Nash allocations are always Walras allocations

We state and prove now our main result which establishes that, in the bilateral oligopoly model described in the previous section, any Cournot-Nash allocation is a Walras allocation.

Theorem 1. *Under Assumptions 1, 2, and 3, let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Then, the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium.*

Proof. Let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. We show that $a_{t1}\hat{\mathbf{x}}^1(t) = a_{t2}\hat{\mathbf{x}}^2(t)$, for each $t \in T_1$. Suppose that $a_{\tau 1}\hat{\mathbf{x}}^1(\tau) \neq a_{\tau 2}\hat{\mathbf{x}}^2(\tau)$, for some $\tau \in T_1$. Moreover, suppose, without loss of generality, that $\mathbf{w}^1(\tau) = 0$ and $\mathbf{w}^2(\tau) > 0$. Let

³In order to save in notation, with some abuse we denote by \mathbf{x} both the function $\mathbf{x}(t)$ and the function $\mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b}))$.

$b'(\tau)$ be a strategy of trader τ such that $b'_{21}(\tau)$ is a solution to the equation

$$a_{\tau 1} b'_{21}(\tau) \frac{\bar{\mathbf{b}}_{12}}{\bar{\mathbf{b}}_{21} - \hat{\mathbf{b}}_{21}(\tau)\mu(\tau) + b'_{21}(\tau)\mu(\tau)} = a_{\tau 2}(\mathbf{w}^2(\tau) - b'_{21}(\tau)),$$

which can be rewritten as

$$\alpha b_{21}'^2(\tau) + \beta b_{21}'(\tau) - \gamma = 0,$$

where

$$\begin{aligned} \alpha &= a_{\tau 2}\mu(\tau), \\ \beta &= a_{\tau 1}\bar{\mathbf{b}}_{12} - a_{\tau 2}\mathbf{w}^2(\tau)\mu(\tau) + a_{\tau 2}(\bar{\mathbf{b}}_{21} - \hat{\mathbf{b}}_{21}(\tau)\mu(\tau)), \end{aligned}$$

and

$$\gamma = a_{\tau 2}\mathbf{w}^2(\tau)(\bar{\mathbf{b}}_{21} - \hat{\mathbf{b}}_{21}(\tau)\mu(\tau)).$$

Then, we have that

$$b'_{21}(\tau) = \frac{-\beta + \sqrt{\beta^2 + 4\alpha\gamma}}{2\alpha}.$$

Suppose that $a_{\tau 1}\hat{\mathbf{x}}^1(\tau) > a_{\tau 2}\hat{\mathbf{x}}^2(\tau)$. Then, we have that

$$a_{\tau 1}\hat{\mathbf{b}}_{21}(\tau) \frac{\bar{\mathbf{b}}_{12}}{\bar{\mathbf{b}}_{21} - \hat{\mathbf{b}}_{21}(\tau)\mu(\tau) + \hat{\mathbf{b}}_{21}(\tau)\mu(\tau)} > a_{\tau 2}(\mathbf{w}^2(\tau) - \hat{\mathbf{b}}_{21}(\tau)).$$

But then, it must be that

$$\alpha\hat{\mathbf{b}}_{21}^2(\tau) + \beta\hat{\mathbf{b}}_{21}(\tau) - \gamma > 0,$$

and this implies that $\hat{\mathbf{b}}_{21}(\tau) > b'_{21}(\tau)$. Then, it is straightforward to verify that

$$\mathbf{x}^2(\tau, b'(\tau), p(\hat{\mathbf{b}} \setminus b'(\tau))) = \mathbf{w}^2(\tau) - b'_{21}(\tau) > \mathbf{w}^2(\tau) - \hat{\mathbf{b}}_{21}(\tau) = \mathbf{x}^2(\tau, \hat{\mathbf{b}}(\tau), p(\hat{\mathbf{b}})).$$

But then, it follows that

$$\begin{aligned} u_{\tau}(\mathbf{x}(\tau, b'(\tau), p(\hat{\mathbf{b}} \setminus b'(\tau)))) &= a_{\tau 2}(\mathbf{w}^2(\tau) - b'_{21}(\tau)) \\ &> a_{\tau 2}(\mathbf{w}^2(\tau) - \hat{\mathbf{b}}_{21}(\tau)) = u_{\tau}(\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), p(\hat{\mathbf{b}}))), \end{aligned}$$

a contradiction. Suppose that $a_{\tau 1}\hat{\mathbf{x}}^1(\tau) < a_{\tau 2}\hat{\mathbf{x}}^2(\tau)$. Then, we have that

$$a_{\tau 1}\hat{\mathbf{b}}_{21}(\tau) \frac{\bar{\mathbf{b}}_{12}}{\bar{\mathbf{b}}_{21} - \hat{\mathbf{b}}_{21}(\tau)\mu(\tau) + \hat{\mathbf{b}}_{21}(\tau)\mu(\tau)} < a_{\tau 2}(\mathbf{w}^2(\tau) - \hat{\mathbf{b}}_{21}(\tau)).$$

But then, it must be that

$$\alpha \hat{\mathbf{b}}_{21}^2(\tau) + \beta \hat{\mathbf{b}}_{21}(\tau) - \gamma < 0,$$

and this implies that $\hat{\mathbf{b}}_{21}(\tau) < b'_{21}(\tau)$. Then, it is straightforward to verify that

$$\begin{aligned} \mathbf{x}^1(\tau, b'(\tau), p(\hat{\mathbf{b}} \setminus b'(\tau))) &= b'_{21}(\tau) \frac{\bar{\mathbf{b}}_{12}}{\bar{\mathbf{b}}_{21} - \hat{\mathbf{b}}_{21}(\tau)\mu(\tau) + b'_{21}(\tau)\mu(\tau)} \\ &> \hat{\mathbf{b}}_{21}(\tau) \frac{\bar{\mathbf{b}}_{12}}{\bar{\mathbf{b}}_{21}} = \mathbf{x}^1(\tau, \hat{\mathbf{b}}(\tau), p(\hat{\mathbf{b}})). \end{aligned}$$

But then, it follows that

$$\begin{aligned} u_\tau(\mathbf{x}(\tau, b'(\tau), p(\hat{\mathbf{b}} \setminus b'(\tau)))) &= a_{\tau 1} b'_{21}(\tau) \frac{\bar{\mathbf{b}}_{12}}{\bar{\mathbf{b}}_{21} - \hat{\mathbf{b}}_{21}(\tau)\mu(\tau) + b'_{21}(\tau)\mu(\tau)} \\ &> a_{\tau 1} \hat{\mathbf{b}}_{21}(\tau) \frac{\bar{\mathbf{b}}_{12}}{\bar{\mathbf{b}}_{21}} = u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), p(\hat{\mathbf{b}}))), \end{aligned}$$

a contradiction. Therefore, we can conclude that $a_{t1}\hat{\mathbf{x}}^1(t) = a_{t2}\hat{\mathbf{x}}^2(t)$, for each $t \in T_1$. This implies that $u_t(\hat{\mathbf{x}}(t)) \geq u_t(y)$ for all $y \in \{x \in R_+^2 : \hat{p}x = \hat{p}\mathbf{w}(t)\}$, as $u_t(x^1, x^2) = \min\{a_{t1}x^1, a_{t2}x^2\}$, for each $t \in T_1$. Moreover, it is straightforward to show (see, for instance, Proposition 3 in Busetto et al. (2013)) that $u_t(\hat{\mathbf{x}}(t)) \geq u_t(y)$ for all $y \in \{x \in R_+^2 : \hat{p}x = \hat{p}\mathbf{w}(t)\}$, for each $t \in T_0$. Hence, the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium. ■

Theorem 1 has the straightforward implication concerning the Pareto optimality properties of a Cournot-Nash allocation established) by the following corollary.

Corollary 1. *Under Assumptions 1, 2, and 3, let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Then, $\hat{\mathbf{x}}$ is Pareto optimal.*

Proof. Let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Then, $\hat{\mathbf{x}}$ is a Walras allocation, by Theorem 1. But then, it is Pareto optimal, by the first fundamental theorem of welfare economics. Hence, a Cournot-Nash allocation $\hat{\mathbf{x}}$ is Pareto optimal. ■

The following example shows that Theorem 1 holds non-vacuously.

Example. Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3. $T_0 = [0, 1]$, $T_1 = \{2, 3\}$, T_0 is taken with Lebesgue measure, $\mu(2) = \mu(3) = 1$, $\mathbf{w}(t) = (4, 0)$, $u_t(x) = \sqrt{x^1} + \sqrt{x^2}$, for each $t \in T_0$, $\mathbf{w}(2) = \mathbf{w}(3) = (0, 4)$, $u_2(x) = u_3(x) = \min\{x^1, x^2\}$. Then, there is a unique Walras allocation, which is also the unique Cournot-Nash allocation.

Proof. The unique Walras equilibrium is the pair (p^*, \mathbf{x}^*) , where $(p^{*1}, p^{*2}) = (2, 1)$, $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (\frac{4}{3}, \frac{16}{3})$, for each $t \in T_0$, $(\mathbf{x}^{*1}(2), \mathbf{x}^{*2}(2)) = (\mathbf{x}^{*1}(3), \mathbf{x}^{*2}(3)) = (\frac{4}{3}, \frac{4}{3})$. The strategy selection \mathbf{b}^* such that $\mathbf{b}_{12}^*(t) = \frac{8}{3}$, for each $t \in T_0$, $\mathbf{b}_{21}^*(2) = \mathbf{b}_{21}^*(3) = \frac{8}{3}$, is a Cournot-Nash equilibrium and $\mathbf{x}^*(t) = \mathbf{x}(t, \mathbf{b}^*(t), p(\mathbf{b}^*))$, for each $t \in T$. Suppose that \mathbf{b}^* is not the unique Cournot-Nash equilibrium. Then, there is a strategy selection \mathbf{b}^{**} such that $\mathbf{b}^{**}(t) \neq \mathbf{b}^*(t)$, for each $t \in V$, where $V \in T_0$ is a coalition, or $\mathbf{b}^{**}(2) \neq \mathbf{b}^*(2)$, or $\mathbf{b}^{**}(3) \neq \mathbf{b}^*(3)$. Then, the allocation \mathbf{x}^{**} such that $\mathbf{x}^{**}(t) = \mathbf{x}(t, \mathbf{b}^{**}(t), p(\mathbf{b}^{**}))$, for each $t \in T$, is a Walras allocation, by Theorem 1, and $\mathbf{x}^{**}(t) \neq \mathbf{x}^*(t)$, for each $t \in V$, or $\mathbf{x}^{**}(2) \neq \mathbf{x}^*(2)$, or $\mathbf{x}^{**}(3) \neq \mathbf{x}^*(3)$, a contradiction. Hence, there is a unique Walras allocation which is also the unique Cournot-Nash allocation. ■

4 Discussion of the model

Codognato et al. (2015) analyzed the relationship between Cournot-Nash and Walras equilibria in the same bilateral oligopoly framework used in this paper, under Assumptions 1, 3, and the two further following assumptions on atoms' utility functions.

Assumption 2'. $u_t : R_+^2 \rightarrow R$ is continuous, strongly monotone, and quasi-concave, for each $t \in T$.

Assumption 4. $u_t : R_+^2 \rightarrow R$ is differentiable, for each $t \in T_1$.⁴

Let us notice that Assumption 2' differs from Assumption 2 in that it imposes that atoms' utility functions are strongly monotone whereas Leon-tievian utility functions introduced in Assumption 2 are only monotone.

Theorem 4 in Codognato et al. (2015) shows that, under Assumptions 1, 2', 3, and 4, a necessary and sufficient condition for a Cournot-Nash

⁴In this assumption, differentiability means continuous differentiability and is to be understood as including the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see, for instance, Kreps (2012), p. 58).

allocation to be a Walras allocation is that all atoms demand a null amount of one of the two commodities. Moreover, their Example 6 shows that their Theorem 4 holds non-vacuously.

On the other hand, our Theorem 1 shows that neither Assumption 2' nor Assumption 4 are necessary conditions for a Cournot-Nash allocation to be Walrasian, since it establishes that, when atoms' utility functions are Leontievidian, as imposed by our Assumption 2, and consequently neither strongly monotone nor differentiable, a Cournot-Nash allocation is always Walrasian.

We state and prove now a proposition which characterizes atoms' assignments at a Cournot-Nash allocation when their utility functions are Leontievidian. Indeed, this proposition establishes that, under Assumptions 1, 2, and 3, at a Cournot-Nash allocation, which is always a Walras allocation by Theorem 1, all atoms demand a strictly positive amount of the two commodities, in the proportion determined by the parameters of their Leontievidian utility function.

Proposition 1. *Under Assumptions 1, 2, and 3, let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Then, $\hat{\mathbf{x}}$ is such that $\hat{\mathbf{x}}(t) \gg 0$, for each $t \in T_1$.*

Proof. Let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Then, $\hat{\mathbf{x}}$ is a Walras allocation, by Theorem 1. We have that $\hat{p} \gg 0$ as the matrix $\hat{\mathbf{B}}$ is irreducible, by Lemma 1 in Sahi and Yao (1989). Consider an atom $\tau \in T_1$ and suppose, without loss of generality, that $\mathbf{w}^1(\tau) = 0$ and $\mathbf{w}^2(\tau) > 0$. We have that $\hat{\mathbf{x}}^1(\tau) = \frac{a_{\tau 2} \hat{p}^2 \mathbf{w}^2(\tau)}{a_{\tau 2} \hat{p}^1 + a_{\tau 1} \hat{p}^2} > 0$ and $\hat{\mathbf{x}}^2(\tau) = \frac{a_{\tau 1} \hat{p}^2 \mathbf{w}^2(\tau)}{a_{\tau 2} \hat{p}^1 + a_{\tau 1} \hat{p}^2} > 0$ as $\hat{\mathbf{x}}$ is a Walras allocation. Hence, $\hat{\mathbf{x}}$ is such that $\hat{\mathbf{x}}(t) \gg 0$, for each $t \in T_1$. ■

This result is explained by the fact that atoms' marginal rate of substitution is not defined when they demand an amount of the two commodities in the fixed proportion determined by the parameters of their Leontievidian utility function, and is either infinite or null, otherwise. As a consequence, under Assumptions 1, 2, and 3, a Cournot-Nash equilibrium cannot occur at a point where atoms' marginal rate of substitution is infinite or null.

Proposition 1 can be compared with a result, for some respect similar, obtained by Codognato et al. (2015) with their Proposition 2, in which they provided a necessary condition for their equivalence theorem to hold when atoms' preferences are represented by an additively separable utility function of the form $u(x) = v^1(x^1) + v^2(x^2)$, for each $x \in R_+^2$.

We repropose here their result, referring to their paper for the proof.

Proposition 2. *Under Assumptions 1, 2', 3, and 4, let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Then, for each $t \in T_1$ such that $u_t(x) = v_t^1(x^1) + v_t^2(x^2)$, $\hat{\mathbf{x}}^1(t) = 0$ only if $-\frac{\partial u_t(0, x^2)}{\partial x^1} / \frac{\partial u_t(0, x^2)}{\partial x^2} > -\infty$, for each $x^2 \in R_+$, and $\hat{\mathbf{x}}^2(t) = 0$ only if $-\frac{\partial u_t(x^1, 0)}{\partial x^1} / \frac{\partial u_t(x^1, 0)}{\partial x^2} < 0$, for each $x^1 \in R_+$.*

This proposition shows that, also when atoms' utility functions are additively separable and satisfy Assumptions 2', 3, and 4, a Walrasian Cournot-Nash allocation of the mixed bilateral version of the Shapley window model cannot occur at a point where the atoms' marginal rate of substitution is infinite or null.

In this regard, it must be noticed that, although the allocations which are at the same time Cournot-Nash and Walrasian are characterized the same way in terms of the marginal rate of substitution when utility functions are Leontievidian and additively separable, a difference emerges concerning the analytical reasons why those allocations are achieved in the two cases: atoms with Leontievidian utility functions achieve an interior Walras assignment as perfect complementarity prevents them from smoothly substituting the two commodities in our framework whereas, atoms with additively separable utility functions obtain corner Walras assignments at a Cournot-Nash equilibrium just because of the substitutability between the two commodities.

As already stressed, both our analysis and that developed by Codognato et al. (2015) is crucially based on the mixed bilateral oligopoly version of the Shapley window model introduced in Section 2.

The Shapley window model is one of the two prototypical market games belonging to the Shapley and Shubik line of research in which markets are complete, i.e., each commodity can be directly exchanged for all the others.

The other prototypical strategic market game with complete markets is that introduced by Amir et al. (1990). In this model, there are a market and a price for each pair of commodities, and the price in each market is determined as the ratio of the total amount of bids in each of the two commodities exchanged in that market.

In general, with more than two commodities, the sets of Cournot-Nash allocations of the two models differ as, in the Shapley window model, a price is determined for each commodity whereas, in the model introduced by Amir et al. (1990), a price is determined for the market of each pair of

commodities and there may be inconsistency between prices corresponding to pairs of markets in which a same commodity is exchanged.

It is straightforward to show that, in bilateral oligopoly, another well-known market game, that introduced by Dubey and Shubik (1978), in which one commodity plays the role of money, can be reduced to the model proposed by Amir et al. (1990), once that one of the two commodities is labeled as money.

Hereafter in this section, we will refer to the mixed bilateral oligopoly version of the Shapley window model as Model 1, and to the mixed bilateral oligopoly version of the model introduced by Amir et al. (1990) as Model 2.

In their Theorem 5, Codognato et al. (2015) showed that, under Assumptions 1, 2', and 3, the sets of Cournot-Nash allocations of Model 1 and Model 2 coincide and that this equivalence extends, *mutatis mutandis*, to the model introduced by Dubey and Shubik (1978). Both Model 1 and Model 2 can then be seen as its possible generalizations.

Since the proof Theorem 5 in Codognato et al. (2015) is crucially based on the assumption that atoms' utility functions are strongly monotone, it cannot be applied to our Leontievan framework.

We address here the question whether an equivalence result like that obtained by Codognato et al. (2015) can be established also in the case where atoms have Leontievan utility functions. If this is possible, the main results obtained in this paper for Model 1 can be extended also to Model 2.

Borrowing from Codognato et al. (2015), we introduce now Model 2 formally. We start with the following definition.

Definition 4. *Given a strategy selection \mathbf{b} , the 2×2 matrix P is said to be the price matrix generated by \mathbf{b} if*

$$p_{ij} = \begin{cases} \frac{\bar{\mathbf{b}}_{ij}}{\bar{\mathbf{b}}_{ji}} & \text{if } \bar{\mathbf{b}}_{ji} \neq 0, \\ 0 & \text{if } \bar{\mathbf{b}}_{ji} = 0, \end{cases}$$

$i, j = 1, 2$.

We denote by $P(\mathbf{b})$ a function which associates with each strategy selection \mathbf{b} the price matrix P generated by \mathbf{b} .

Given a strategy selection \mathbf{b} and a price matrix P , consider the assignment determined as follows:

$$\mathbf{x}^j(t, \mathbf{b}(t), P) = \mathbf{w}^j(t) - \sum_{i=1}^2 \mathbf{b}_{ji}(t) + \sum_{i=1}^2 \mathbf{b}_{ij}(t)p_{ji},$$

$j = 1, 2$, for each $t \in T$.

Given a strategy selection \mathbf{b} and the function $P(\mathbf{b})$, the traders' final holdings are determined according to this rule and consequently expressed by the assignment

$$\mathbf{x}(t) = \mathbf{x}(t, \mathbf{b}(t), P(\mathbf{b})),$$

for each $t \in T$.⁵ It is straightforward to show that this assignment is an allocation.

Then, a Cournot-Nash equilibrium for Model 2 can be defined as follows.

Definition 5. *A strategy selection $\tilde{\mathbf{b}}$ such that $\tilde{\mathbf{B}}$ is irreducible is a Cournot-Nash equilibrium if*

$$u_t(\mathbf{x}(t, \tilde{\mathbf{b}}(t), P(\tilde{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, \tilde{\mathbf{b}} \setminus b(t), P(\tilde{\mathbf{b}} \setminus b(t)))),$$

for each $b(t) \in \mathbf{B}(t)$ and for each $t \in T$.⁶

A Cournot-Nash allocation of Model 2 is an allocation $\tilde{\mathbf{x}}$ such that $\tilde{\mathbf{x}}(t) = \mathbf{x}(t, \tilde{\mathbf{b}}(t), P(\tilde{\mathbf{b}}))$, for each $t \in T$, where $\tilde{\mathbf{b}}$ is a Cournot-Nash equilibrium of Model 2.

The following lemma establishes a relation between prices and hence traders' final holdings of the two models for strategy selections whose aggregate matrices are irreducible. It was proved by Codognato et al. (2015) (see the online appendix).

Lemma. *If \mathbf{b} is a strategy selection such that $\bar{\mathbf{B}}$ is irreducible, then $\frac{p^i(\mathbf{b})}{p^j(\mathbf{b})} = p_{ji}(\mathbf{b})$, $i, j = 1, 2$, and $\mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b})) = \mathbf{x}(t, \mathbf{b}(t), P(\mathbf{b}))$, for each $t \in T$.*

We are now able to prove the following theorem, which establishes an equivalence between the sets of Cournot-Nash allocations of Model 1 and Model 2 when atoms have Leontieviaan utility functions.

Theorem 2. *Under Assumptions 1, 2, and 3, the sets of Cournot-Nash allocations of Model 1 and Model 2 coincide.*

Proof. Let $\hat{\mathbf{x}}$ be a Cournot-Nash allocation of Model 1. Then, there is a strategy selection $\hat{\mathbf{b}}$ which is a Cournot-Nash equilibrium of Model 1 and is

⁵In order to save in notation, with some abuse we denote by \mathbf{x} both the function $\mathbf{x}(t)$ and the function $\mathbf{x}(t, \mathbf{b}(t), P(\mathbf{b}))$.

⁶According to Amir et al. (1990), the market for commodities 1 and 2 is active if $\bar{\mathbf{b}}_{12} > 0$ and $\bar{\mathbf{b}}_{21} > 0$ and then if and only if $\bar{\mathbf{B}}$ is irreducible. Therefore, as this definition of a Cournot-Nash equilibrium explicitly refers to irreducible matrices, it applies only to Cournot-Nash equilibria at which the market for commodities 1 and 2 is active.

such that $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Suppose that $\hat{\mathbf{x}}$ is not a Cournot-Nash allocation of Model 2. Then, there exists a trader $\tau \in T$ and a strategy $b(\tau) \in \mathbf{B}(\tau)$ such that

$$u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}} \setminus b(\tau), P(\hat{\mathbf{b}} \setminus b(\tau)))) > u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), P(\hat{\mathbf{b}}))).$$

We have that $\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), p(\hat{\mathbf{b}})) = \mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), P(\hat{\mathbf{b}}))$, by the Lemma, as $\hat{\mathbf{B}}$ is irreducible. Suppose that the matrix $\hat{\mathbf{B}} \setminus b(\tau)$ is irreducible. Then, $\mathbf{x}(\tau, \hat{\mathbf{b}} \setminus b(\tau), p(\hat{\mathbf{b}} \setminus b(\tau))) = \mathbf{x}(\tau, \hat{\mathbf{b}} \setminus b(\tau), P(\hat{\mathbf{b}} \setminus b(\tau)))$, by the Lemma. But then,

$$u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}} \setminus b(\tau), p(\hat{\mathbf{b}} \setminus b(\tau)))) > u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), p(\hat{\mathbf{b}}))),$$

a contradiction. Suppose that the matrix $\hat{\mathbf{B}} \setminus b(\tau)$ is not irreducible. Then, we must have that $\tau \in T_1$ as $\hat{\mathbf{B}} \setminus b(t) = \hat{\mathbf{B}}$, for each $t \in T_0$. Assume, without loss of generality, that $\mathbf{w}^1(\tau) = 0$ and $\mathbf{w}^2(\tau) > 0$. Then, we must have that $\hat{\mathbf{b}}_{21}(\tau) = \tilde{\mathbf{b}}_{21}$ as the matrix $\hat{\mathbf{B}} \setminus b(\tau)$ is not irreducible. But then, we have that $\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), p(\hat{\mathbf{b}})) = (\hat{\mathbf{b}}_{12}, \mathbf{w}^2(\tau) - \hat{\mathbf{b}}_{21}(\tau))$ and $\mathbf{x}(\tau, \hat{\mathbf{b}} \setminus b(\tau), p(\hat{\mathbf{b}} \setminus b(\tau))) = (0, \mathbf{w}^2(\tau))$. This implies that

$$u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}} \setminus b(\tau), p(\hat{\mathbf{b}} \setminus b(\tau)))) = 0 > u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), p(\hat{\mathbf{b}}))) \geq 0,$$

a contradiction. Therefore, $\hat{\mathbf{x}}$ is a Cournot-Nash allocation of Model 2. Let $\tilde{\mathbf{x}}$ be a Cournot-Nash allocation of Model 2. Then, there is a strategy selection $\tilde{\mathbf{b}}$ which is a Cournot-Nash equilibrium of Model 2 and is such that $\tilde{\mathbf{x}}(t) = \mathbf{x}(t, \tilde{\mathbf{b}}(t), P(\tilde{\mathbf{b}}))$, for each $t \in T$. We have that $\mathbf{x}(t, \tilde{\mathbf{b}}(\tau), P(\tilde{\mathbf{b}})) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$, by the Lemma, as $\hat{\mathbf{B}}$ is irreducible. But then, we must have that $a_{i1}\tilde{\mathbf{x}}^1(t) = a_{i2}\tilde{\mathbf{x}}^2(t)$, for each $t \in T_1$, by the same argument used in the proof of Theorem 1. Suppose that $\tilde{\mathbf{x}}$ is not a Cournot-Nash allocation of Model 1. Then, the previous argument leads, *mutatis mutandis*, to the same kind of contradictions. Therefore, $\tilde{\mathbf{x}}$ is a Cournot-Nash allocation of Model 1. Hence, the sets of Cournot-Nash allocations of Model 1 and Model 2 coincide. ■

Theorem 2 has the following corollary, showing that our Theorem 1 extends, *mutatis mutandis*, to Model 2.

Corollary 2. *Under Assumptions 1, 2, 3, let $\tilde{\mathbf{b}}$ be a Cournot-Nash equilibrium of Model 2 and let $\tilde{p} = (\tilde{\mathbf{b}}_{21}, \tilde{\mathbf{b}}_{12})$ and $\tilde{\mathbf{x}}(t) = \mathbf{x}(t, \tilde{\mathbf{b}}(t), p(\tilde{\mathbf{b}}))$, for each $t \in T$. Then, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium.*

Proof. Let $\tilde{\mathbf{b}}$ be a Cournot-Nash equilibrium of Model 2 and let $\tilde{p} = (\tilde{\mathbf{b}}_{21}, \tilde{\mathbf{b}}_{12})$ and $\tilde{\mathbf{x}}(t) = \mathbf{x}(t, \tilde{\mathbf{b}}(t), p(\tilde{\mathbf{b}}))$, for each $t \in T$. $\tilde{\mathbf{b}}$ is a Cournot-Nash

equilibrium of Model 1, by Theorem 2, and $\tilde{p} = (\tilde{\mathbf{b}}_{21}, \tilde{\mathbf{b}}_{12}) = (p^1(\tilde{\mathbf{b}}), p^2(\tilde{\mathbf{b}}))$, by Definition 2. Hence, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium, by Theorem 1. ■

Finally, let us stress that, for the reasons exposed above, our Theorem 1 also extends, *mutatis mutandis*, to the model proposed by Dubey and Shubik (1978).

5 Conclusion

The main result of this paper is that, in the framework of a mixed bilateral oligopoly, when atoms have Leontievan utility functions, Cournot-Nash allocations are always Walras allocations. In a further step of our research, we propose to extend our analysis, studying whether an equivalence between the set of Cournot-Nash and Walras allocations can be obtained also for exchange economies with more than two commodities. We could then address some further issues including the computational ones considered for finite exchange economies with Leontievan traders by Ye (2007) and Codenotti et al. (2008), among others, in a framework related to operations research. This might also open the way for an investigation of economies with production where firms have Leontievan technologies. Finally, we remind that the equivalence theorem between the core and the set of Walras allocations proved by Shitovitz (1973) rests on the assumptions that atoms' preferences are strongly monotone. Here, we have proved a noncooperative equivalence result which holds when atoms' preferences are monotone but not necessarily strongly monotone. Our results should stimulate a further investigation on the validity of the core equivalence theorems beyond the strong monotonicity assumption.

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