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Appendix to: Efficient European and American Option Pricing Under a Jump-diffusion Process

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Appendix to: Efficient European and American Option Pricing Under a Jump-diffusion Process

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Abstract

This paper constitutes the Appendix of the article "Efficient European and American option pricing under a jump-diffusion process". Here are detailed the proofs that could not be part of the main sections of the article, for length and readability reasons. Every section is dedicated to a proof, starts with the recollection of the statement of the lemma, proposition or theorem involved and continues with its proof.

1. Proof of Lemma 5.10

Lemma:

$$Q_N^{\bar{k}}(k) \leq \sum_{i=1}^{N} \widetilde{Q}_N(2\bar{k} - k + 2i)$$

$$Q_{N\bar{l}}(k) \leq \sum_{i=1}^{N} \widetilde{Q}_N(2\bar{l} + k + 2i)$$

for all $-\bar{l} \leq k \leq \bar{k}$.

Proof:

The proof is analogous to that of Lemma 5.4. When the original path first trespasses the $\bar{k}$ level, it can reach level $\bar{k} + 1, \ldots, \bar{k} + N$. Therefore its reflection (defined as in Lemma 5.4) can end at level $2\bar{k} - k + 2$, $2\bar{k} - k + 4 \ldots, 2\bar{k} - k + 2N$. Likewise for the paths that cross the $-\bar{l}$ level.

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2. Proof of Proposition 5.11

Proposition: *Given* G *and* W_N *as defined in Equation (4.3) in the main article, for integers* $0 \leq k, \bar{k} \leq Nn$ *we have:*

$$\text{For } k \geq N\lceil 2W_N - 1\rceil \qquad \widetilde{Q}_N(k) \leq G\frac{W_N^{\left\lfloor \frac{k}{N} \right\rfloor}}{\left\lfloor \frac{k}{N} \right\rfloor!} \tag{1}$$

$$\text{For } \bar{k} \geq N\lceil 2W_N - 1\rceil \qquad \sum_{k=\bar{k}}^{Nn} \widetilde{Q}_N(k) \leq 2GN\frac{W_N^{\left\lfloor \frac{\bar{k}}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}}{N} \right\rfloor!} \tag{2}$$

$$\text{For } \bar{k} \geq N\lceil 2e^{Nh}W_N - 1\rceil \qquad \sum_{k=\bar{k}}^{Nn} e^{hk}\widetilde{Q}_N(k) \leq 2G\frac{(e^{hN}W_N)^{\left\lfloor \frac{\bar{k}}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}}{N} \right\rfloor!}\sum_{r=0}^{N-1} e^{hr} \tag{3}$$

$$\text{For } \bar{k} \geq N\lceil 2W_N - 1\rceil \qquad \sum_{k=\bar{k}}^{Nn} e^{-hk}\widetilde{Q}_N(-k) \leq 2G\frac{(e^{-hN}W_N)^{\left\lfloor \frac{\bar{k}}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}}{N} \right\rfloor!}\sum_{r=0}^{N-1} e^{-hr} \tag{4}$$

Proof:

We need an upper estimate of the probability $Q_N(k)$ of reaching level $k \geq 0$ in the jump dynamics. This will allow us to obtain an upper estimate of how much the value of the option in (n, j, k) for some j contributes to the current value.

We recall that for a fixed N, in a single timestep Δt the possible jump moves are $-Nh, \ldots, -h, 0, h, \ldots, Nh$. For simplicity, in the following we will talk about $-N, \ldots, -1, 0, 1, \ldots, N$ jumps.

Level $k \geq 0$ at maturity can be reached with a variety of possible combinations of jumps. In order to consider all the possible paths that arrive at level k in n timesteps, exactly as we did in the $N = 1$ case, we distinguish between the positive and the negative jumps: if $k \geq 0$ is the total balance and the sum of all negative jumps is $-l$, then the sum of all positive jumps must be $k + l$, with $l \geq 0$. $Q_N(k)$ is the sum of all probabilities of reaching balance level k with a negative balance of $-l$, over all possible non negative l, subject to the condition of a total of n moves.

Let us denote by e_j^- the number of $-j$ jumps and e_j^+ the number of j jumps in a path, for $j = 1, \ldots, N$.

With this notation, the probability $Q_N(k)$ of reaching at maturity level $k \geq 0$ for the jump dynamics is given by:

$$Q_N(k) = \sum_l \sum_{e_N^+} \cdots \sum_{e_1^+} \sum_{e_N^-} \cdots \sum_{e_1^-} C(e_N^+, e_N^-, e_{N-1}^+, e_{N-1}^-, \ldots, e_1^+, e_1^-)q_{+N}^{e_N^+} \cdots q_{+1}^{e_1^+} q_{-N}^{e_N^-} \cdots q_{-1}^{e_1^-} q_0^{e_0}$$

where the e_0 exponent is given by $n - \sum_{i=1}^N e_i^+ - \sum_{i=1}^N e_i^-$, and $C(e_N^+, e_N^-, e_{N-1}^+, e_{N-1}^-, \ldots, e_1^+, e_1^-)$ denotes the number of combinations of the n factors, once the exponents are fixed, and is equal to

$$C(e_N^+, e_N^-, e_{N-1}^+, e_{N-1}^-, \ldots, e_1^+, e_1^-) = \frac{n!}{e_N^+! e_N^-! e_{N-1}^+! e_{N-1}^-! \ldots e_1^+! e_1^-! e_0!}.$$

While in the $N = 1$ setting, a $-l$ negative balance meant l jumps of the $-h$ kind, and similarly a $k + l$ positive balance meant $k + l$ jumps of the $+h$ kind, here the situation is complicated by the possibility of different jump amplitudes, so extra care is needed in order to understand the relation between l and the exponents e_i^+, e_i^-.

We use Euclidean division in order to write l as a multiple of N plus a remainder $0 \leq r_N^- \leq N - 1$: $l = Nz + r_N^-$. This means that the negative balance $-l$ is due to at most z jumps of the $-N$ kind, and the difference between Nz and l shall be covered with smaller jumps.

Instead of summing over all possible l, then, it will be easier to consider the summation over all possible z and $0 \leq r_N^- \leq N - 1$.

For any fixed z and r_N^-, we will have at most z jumps of the $-N$ kind, therefore we need to vary e_N^- between 0 and z; the choice of e_N^- sets additional constraints for e_{N-1}^-, and proceeding backwards the choice of every e_i^- sets additional constraints for e_{i-1}^-. We apply the same idea to the positive balance $k + l$: given k, the values t and $0 \leq r_N \leq N - 1$ such that $k = Nt + r_N$ are uniquely determined; therefore for any given pair of z and r_N^- the positive balance can be written as $N(t + z) + r_N + r_N^-$. This provides the limitation for e_N^+, and the choice of every e_j^+ imposes further conditions on the possible values for e_{j-1}^+.

In order to better express the relationships and mutual limitations between exponents, we need a change in perspective in the summations.

For any fixed z, let us define $b_{N-1} = z - e_N^-$. Of the negative balance $-(Nz + r_N^-)$, then, $-Ne_N^-$ will be covered by $-N$ jumps and the rest, $-(Nb_{N-1} + r_N^-)$, by jumps of smaller amplitude. Instead of summing over e_N^- from 0 to z, we sum over b_{N-1}, that is over how many of the $-Nz$ are covered by jumps of amplitude smaller than N.

Once fixed z, r_N^- and e_N^-, we have a negative balance of $-(Nb_{N-1} + r_N^-)$ to cover with negative jumps of amplitude at most $N - 1$: we compute the Euclidean division of $Nb_{N-1} + r_N^-$ by $N - 1$: the quotient $z_{N-1} = \lfloor \frac{Nb_{N-1}+r_N^-}{N-1} \rfloor$ is an upper bound (we shall consider the more stringent between this value and the condition of a total of n moves), and we call r_{N-1}^- the remainder. Once again, instead of summing over e_{N-1}^-, we sum over $b_{N-2} = z_{N-1} - e_{N-1}^-$.

We repeatedly use Euclidean division in order to find the upper bounds for all e_j^-, and operate in the same way for the positive jumps, where we similarly introduce the a_j and r_j^+ values.

The probability $Q_N(k)$ of reaching at maturity level $k \geq 0$ for the jump dynamics can then be written as:

$$Q_N(k) = \sum_{r_N^-=0}^{N-1} \sum_{z} \sum_{a_{N-1}} \cdots \sum_{a_1} \sum_{b_{N-1}} \cdots \sum_{b_1} \frac{n!}{e_N^+! e_N^-! e_{N-1}^+! e_{N-1}^-! \dots e_1^+! e_1^-! e_0!} q_{+N}^{e_N^+} \cdots q_{+1}^{e_1^+} q_{-N}^{e_N^-} \cdots q_{-1}^{e_1^-} q_0^{e_0}.$$

The indices a_j (b_j) are indicators of how much of the total positive (respectively, negative) balance is due to moves of amplitude at most j, and are related to the exponents in the following way:

$$e_N^- = z - b_{N-1} \qquad e_N^+ = t + z + \left\lfloor \frac{r_N + r_N^-}{N} \right\rfloor - a_{N-1}$$

$$e_i^- = \left\lfloor \frac{(i+1)b_i + r_{i+1}^-}{i} \right\rfloor - b_{i-1} \text{ where } r_i^- \text{ is the remainder of } \frac{(i+1)b_i + r_{i+1}^-}{i} \text{ for } 1 < i < N$$

$$e_i^+ = \left\lfloor \frac{(i+1)a_i + r_{i+1}^+}{i} \right\rfloor - a_{i-1} \text{ where } r_i^+ \text{ is the remainder of } \frac{(i+1)a_i + r_{i+1}^+}{i} \text{ for } 1 < i < N$$

$$e_1^- = 2b_1 + r_2^- \qquad e_1^+ = 2a_1 + r_2^+$$

Substituting $c_{\pm i}$ with w_i, we obtain

$$\widetilde{Q}_N(k) = \sum_{r_N^-=0}^{N-1} \sum_{z} \sum_{a_{N-1}} \cdots \sum_{a_1} \sum_{b_{N-1}} \cdots \sum_{b_1} \frac{n!}{e_N^+! e_N^-! e_{N-1}^+! e_{N-1}^-! \dots e_1^+! e_1^-! e_0!} \frac{w_N^{e_N^+} \cdots w_1^{e_1^+} w_N^{e_N^-} \cdots w_1^{e_1^-}}{n^{\sum_{i=1}^N e_i^+ + \sum_{i=1}^N e_i^-}} q_0^{e_0}$$

Since $q_0 \leq 1$ and $\frac{n!}{e_0! n^{\sum_{i=1}^N e_i^+ + \sum_{i=1}^N e_i^-}} \leq 1$:

$$\widetilde{Q}_N(k) \leq \sum_{r_N^-=0}^{N-1} \sum_{z} \sum_{a_{N-1}} \cdots \sum_{a_1} \frac{w_N^{e_N^+} \cdots w_1^{e_1^+}}{e_N^+! e_{N-1}^+! \dots e_1^+!} \sum_{b_{N-1}} \cdots \sum_{b_1} \frac{w_N^{e_N^-} \cdots w_1^{e_1^-}}{e_N^-! e_{N-1}^-! \dots e_1^-!}$$

We treat separately the positive and the negative parts, and we work from the inside outwards.

$$\sum_{b_{N-1}} \frac{w_N^{e_N^-}}{e_N^-!} \cdots \sum_{b_3} \frac{w_4^{e_4^-}}{e_4^-!} \sum_{b_2} \frac{w_3^{e_3^-}}{e_3^-!} \sum_{b_1} \frac{w_2^{e_2^-}}{e_2^-!} \frac{w_1^{e_1^-}}{e_1^-!} =$$

$$= \sum_{b_{N-1}} \frac{w_N^{e_N^-}}{e_N^-!} \cdots \sum_{b_3} \frac{w_4^{e_4^-}}{e_4^-!} \sum_{b_2} \frac{w_3^{e_3^-}}{e_3^-!} \sum_{b_1} \frac{w_2^{\left\lfloor \frac{3b_2 + r_3^-}{2} \right\rfloor - b_1}}{\left(\left\lfloor \frac{3b_2 + r_3^-}{2} \right\rfloor - b_1 \right)!} \frac{w_1^{2b_1 + r_2^-}}{(2b_1 + r_2^-)!}$$

$$\leq \sum_{b_{N-1}} \frac{w_N^{e_N^-}}{e_N^-!} \cdots \sum_{b_3} \frac{w_4^{e_4^-}}{e_4^-!} \sum_{b_2} \frac{w_3^{e_3^-}}{e_3^-!} w_1^{r_2^-} \frac{(w_2 + w_1^2)^{\left\lfloor \frac{3b_2 + r_3^-}{2} \right\rfloor}}{\left\lfloor \frac{3b_2 + r_3^-}{2} \right\rfloor !}$$

Since r_2^- is the remainder of $\frac{3b_2+r_3^-}{2}$, it can only assume the values 0 or 1; therefore we can write:

$$\sum_{b_{N-1}} \frac{w_N^{e_N^-}}{e_N^-!} \cdots \sum_{b_3} \frac{w_4^{e_4^-}}{e_4^-!} \sum_{b_2} \frac{w_3^{e_3^-}}{e_3^-!} w_1^{r_2^-} \frac{(w_2 + w_1^2)^{\left\lfloor \frac{3b_2+r_3^-}{2} \right\rfloor}}{\left\lfloor \frac{3b_2+r_3^-}{2} \right\rfloor !} \leq$$

$$\leq \sum_{b_{N-1}} \frac{w_N^{e_N^-}}{e_N^-!} \cdots \sum_{b_3} \frac{w_4^{e_4^-}}{e_4^-!} \sum_{b_2} \frac{w_3^{\left\lfloor \frac{4b_3+r_4^-}{3} \right\rfloor - b_2}}{(\left\lfloor \frac{4b_3+r_4^-}{3} \right\rfloor - b_2)!} \max\{w_1, 1\} \frac{(w_2 + w_1^2)^{\frac{3b_2+r_3^- - r_2^-}{2}}}{\frac{3b_2+r_3^- - r_2^-}{2}\,!}$$

According to the definitions in Equation (4.3) in the main article, $\max\{w_1, 1\} = \max\{W_1^1, W_1^0\} = M_1$, and $w_2 + w_1^2 = W_2$.

In general, we take care of the sum over b_{i-1}, for $1 < i < N$, in the following way:

$$\sum_{b_{i-1}} \frac{w_i^{\left\lfloor \frac{(i+1)b_i+r_{i+1}^-}{i} \right\rfloor - b_{i-1}}}{\left(\left\lfloor \frac{(i+1)b_i+r_{i+1}^-}{i} \right\rfloor - b_{i-1}\right)!} \frac{W_{i-1}^{\left\lfloor \frac{ib_{i-1}+r_i^-}{i-1} \right\rfloor}}{\left\lfloor \frac{ib_{i-1}+r_i^-}{i-1} \right\rfloor !} =$$

$$= \sum_{b_{i-1}} \frac{w_i^{\left\lfloor \frac{(i+1)b_i+r_{i+1}^-}{i} \right\rfloor - b_{i-1}}}{(\left\lfloor \frac{(i+1)b_i+r_{i+1}^-}{i} \right\rfloor - b_{i-1})!} \frac{W_{i-1}^{\frac{ib_{i-1}+r_i^- - r_{i-1}^-}{i-1}}}{\left\lfloor \frac{ib_{i-1}+r_i^-}{i-1} \right\rfloor !} \leq$$

$$\leq \sum_{b_{i-1}} \frac{w_i^{\left\lfloor \frac{(i+1)b_i+r_{i+1}^-}{i} \right\rfloor - b_{i-1}}}{(\left\lfloor \frac{(i+1)b_i+r_{i+1}^-}{i} \right\rfloor - b_{i-1})!} \frac{(W_{i-1}^{\frac{i}{i-1}})^{b_{i-1}}}{b_{i-1}!} W_{i-1}^{\frac{r_i^- - r_{i-1}^-}{i-1}} \leq$$

$$\leq \sum_{b_{i-1}} \frac{w_i^{\left\lfloor \frac{(i+1)b_i+r_{i+1}^-}{i} \right\rfloor - b_{i-1}}}{(\left\lfloor \frac{(i+1)b_i+r_{i+1}^-}{i} \right\rfloor - b_{i-1})!} \frac{(W_{i-1}^{\frac{i}{i-1}})^{b_{i-1}}}{b_{i-1}!} \max\{W_{i-1}, W_{i-1}^{-\frac{i-2}{i-1}}\} =$$

$$= M_{i-1} \frac{(w_i + W_{i-1}^{\frac{i}{i-1}})^{\left\lfloor \frac{(i+1)b_i+r_{i+1}^-}{i} \right\rfloor}}{\left\lfloor \frac{(i+1)b_i+r_{i+1}^-}{i} \right\rfloor !} = M_{i-1} \frac{W_i^{\left\lfloor \frac{(i+1)b_i+r_{i+1}^-}{i} \right\rfloor}}{\left\lfloor \frac{(i+1)b_i+r_{i+1}^-}{i} \right\rfloor !}$$

and similarly for the sum over a_{i-1}, for $2 \leq i < N$. Proceeding in this way for both the negative and the positive balance parts of the summation, we get

$$
\begin{aligned}
\widetilde{Q}_N(k) &\leq \prod_{j=1}^{N-1} M_j^2 \sum_{r_N^-=0}^{N-1} \sum_z \frac{W_N^z}{z!} \frac{W_N^{t+z+\left\lfloor \frac{r_N+r_N^-}{N} \right\rfloor}}{(t+z+\left\lfloor \frac{r_N+r_N^-}{N} \right\rfloor)!} \\
&\leq \prod_{j=1}^{N-1} M_j^2 \sum_z \frac{W_N^z}{z!} \frac{W_N^{t+z}}{(t+z)!} \sum_{r_N^-=0}^{N-1} W_N^{\left\lfloor \frac{r_N+r_N^-}{N} \right\rfloor} \\
&\leq \prod_{j=2}^{N-1} M_j^2 \sum_z \frac{W_N^z}{z!} \sum_z \frac{W_N^{t+z}}{(t+z)!} N \max\{W_N, 1\} \\
&\leq N \max\{W_N, 1\} \prod_{j=2}^{N-1} M_j^2 e^{W_N} \cdot 2 \frac{W_N^t}{t!}
\end{aligned}
$$

for $t \geq 2W_N - 1$. Calling $G = 2N \max\{W_N, 1\} \prod_{j=2}^{N-1} M_j^2 e^{W_N}$ we have Equation (1) for $k \geq N\lceil 2W_N - 1 \rceil$.

Now we apply the previous inequality to the summation $\sum_{k=\bar{k}}^{Nn} \widetilde{Q}_N(k)$, obtaining

$$
\begin{aligned}
\sum_{k=\bar{k}}^{+\infty} \widetilde{Q}_N(k) &\leq \sum_{k=\bar{k}}^{+\infty} G \frac{W_N^{\left\lfloor \frac{k}{N} \right\rfloor}}{\left\lfloor \frac{k}{N} \right\rfloor !} \\
&\leq 2GN \frac{W_N^{\left\lfloor \frac{\bar{k}}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}}{N} \right\rfloor !}
\end{aligned}
$$

provided that $\bar{k} \geq N\lceil 2W_N - 1 \rceil$.

We apply again Equation (1) to the summation $\sum_{k=\bar{k}}^{+\infty} e^{hk} \widetilde{Q}_N(k)$; for $\bar{k} \geq N\lceil 2e^{Nh} W_N - 1 \rceil$ we have:

$$
\sum_{k=\bar{k}}^{+\infty} e^{hk} \widetilde{Q}_N(k) \leq \sum_{k=\bar{k}}^{Nn} e^{hk} G \frac{W_N^{\left\lfloor \frac{k}{N} \right\rfloor}}{\left\lfloor \frac{k}{N} \right\rfloor !} \leq G \sum_{t=\left\lfloor \frac{\bar{k}}{N} \right\rfloor}^{+\infty} \sum_{r=0}^{N-1} e^{hNt+hr} \frac{W_N^t}{t!} \leq 2G \frac{(e^{hN} W_N)^{\left\lfloor \frac{\bar{k}}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}}{N} \right\rfloor !} \sum_{r=0}^{N-1} e^{hr}. \tag{5}
$$

Similarly, we obtain the analogous inequality for $\sum_{k=\bar{k}}^{+\infty} e^{-hk} \widetilde{Q}_N(-k)$ with $\bar{k} \geq N\lceil 2W_N - 1 \rceil$.

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3. Proof of Theorem 4.1

Theorem: *Given* $\varepsilon > 0$*, considering* V *the HS European call option value, taking*

$$
\bar{k} \geq \max\{N \left\lceil e^{hN+1} W_N - \ln \varepsilon + \ln(4S_0 G) + (\alpha - r)\tau + \ln k_+ \right\rceil - 1, N \left\lceil 2e^{hN} W_N - 1 \right\rceil - 1\} \tag{6}
$$

$$
\bar{l} \geq \max\{N \left\lceil e^{-hN+1} W_N - \ln \varepsilon + \ln(4S_0 G) + (\alpha - r)\tau + \ln k_- \right\rceil - 1, N \left\lceil 2e^{hN} W_N - 1 \right\rceil - 1\} \tag{7}
$$

with k_+ and k_- the following constants,

$$k_+ = \sum_{r=0}^{N-1} e^{hr} + N\max\{W_N^2, 1\}e^{2hN} \sum_{r=0}^{N-1} e^{-hr}$$
$$k_- = \sum_{r=0}^{N-1} e^{-hr} + N\max\{W_N^2, 1\} \sum_{r=0}^{N-1} e^{hr},$$

we have that the European call option value V^{TT} obtained via truncation of the tree at levels $\bar{k}$ and $-\bar{l}$ satisfies:

$$V - V^{TT} < \varepsilon.$$

Proof:

Combining Equation (5.3) in the main article,

$$V - V^{PT} \leq e^{(\alpha-r)\tau} S_0 \left(\sum_{k=\bar{k}+1}^{Nn} e^{hk} \widetilde{Q}_N(k) + \sum_{k=\bar{l}+1}^{Nn} e^{-hk} \widetilde{Q}_N(k) \right)$$

and Equation (5.7) in the main article, to which we apply Lemma 5.10,

$$\begin{aligned} V^{PT} - V^{TT} &\leq e^{(\alpha-r)\tau} S_0 \sum_{k=-\bar{l}}^{\bar{k}} e^{hk} (Q_N^{\bar{k}}(k) + Q_{N\bar{l}}(k)) \\ &\leq e^{(\alpha-r)\tau} S_0 \left(\sum_{s=\bar{k}+2}^{2\bar{k}+\bar{l}+2} e^{h(2\bar{k}-s+2)} \sum_{i=0}^{N-1} \widetilde{Q}_N(s+2i) + \sum_{s=\bar{l}+2}^{2\bar{l}+\bar{k}+2} e^{h(s-2\bar{l}-2)} \sum_{i=0}^{N-1} \widetilde{Q}_N(s+2i) \right) \end{aligned}$$

the difference between V and V^{TT} is less or equal than the sum of four discarded parts:

$$V - V^{TT} \leq e^{(\alpha-r)\tau} S_0 \left(\sum_{k=\bar{k}+1}^{Nn} e^{hk} \widetilde{Q}_N(k) + \sum_{k=\bar{l}+1}^{Nn} e^{-hk} \widetilde{Q}_N(k) + e^{h(2\bar{k}+2)} \sum_{s=\bar{k}+2}^{Nn} e^{-hs} \sum_{i=0}^{N-1} \widetilde{Q}_N(s+2i) + e^{h(-2\bar{l}-2)} \sum_{s=\bar{l}+2}^{Nn} e^{hs} \sum_{i=0}^{N-1} \widetilde{Q}_N(s+2i) \right)$$

By Proposition 5.11:

$$V - V^{TT} \leq e^{(\alpha-r)\tau} S_0 G \left(2 \frac{(e^{hN} W_N)^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor !} \sum_{r=0}^{N-1} e^{hr} + 2 \frac{(e^{-hN} W_N)^{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor !} \sum_{r=0}^{N-1} e^{-hr} + \right. \tag{8}$$

$$\left. + e^{h(2\bar{k}+2)} \sum_{s=\bar{k}+2}^{Nn} e^{-hs} \sum_{i=0}^{N-1} \frac{W_N^{\left\lfloor \frac{s+2i}{N} \right\rfloor}}{\left\lfloor \frac{s+2i}{N} \right\rfloor !} + e^{h(-2\bar{l}-2)} \sum_{s'=\bar{l}+2}^{Nn} e^{hs'} \sum_{i=0}^{N-1} \frac{W_N^{\left\lfloor \frac{s'+2i}{N} \right\rfloor}}{\left\lfloor \frac{s'+2i}{N} \right\rfloor !} \right) \tag{9}$$

where we operated the substitutions $s = 2\bar{k} - k + 2$, $s' = 2\bar{l} + k + 2$ and $G = 2N\max\{W_N, 1\}e^{W_N}\prod_{i=1}^{N-1} M_i^2$, and considered $\bar{k} \geq N\lceil 2e^{hN}W_N - 1\rceil - 1$ and $\bar{l} \geq N\lceil 2W_N - 1\rceil - 1$.

Since $\left\lfloor \frac{s}{N} \right\rfloor \leq \left\lfloor \frac{s+2i}{N} \right\rfloor \leq \left\lfloor \frac{s}{N} \right\rfloor + 2$ for $0 \leq i < N$, we have that $\frac{W_N^{\left\lfloor \frac{s+2i}{N} \right\rfloor}}{\left\lfloor \frac{s+2i}{N} \right\rfloor!} \leq \frac{W_N^{\left\lfloor \frac{s}{N} \right\rfloor}}{\left\lfloor \frac{s}{N} \right\rfloor!} \cdot \max\{W_N^2, 1\}$:

$$
\begin{aligned}
V - V^{TT} \leq & 2e^{(\alpha-r)\tau}S_0 G\left(\frac{(e^{hN}W_N)^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor!}\sum_{r=0}^{N-1} e^{hr} + \frac{(e^{-hN}W_N)^{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor!}\sum_{r=0}^{N-1} e^{-hr}\right) + \\
& + e^{(\alpha-r)\tau}S_0 GN\max\{W_N^2, 1\}\left(e^{h(2\bar{k}+2)}\sum_{s=\bar{k}+2}^{Nn} e^{-hs}\frac{W_N^{\left\lfloor \frac{s}{N} \right\rfloor}}{\left\lfloor \frac{s}{N} \right\rfloor!} + e^{h(-2\bar{l}-2)}\sum_{s=\bar{l}+2}^{Nn} e^{hs}\frac{W_N^{\left\lfloor \frac{s}{N} \right\rfloor}}{\left\lfloor \frac{s}{N} \right\rfloor!}\right) \\
\leq & 2e^{(\alpha-r)\tau}S_0 G\left(\frac{(e^{hN}W_N)^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor!}\sum_{r=0}^{N-1} e^{hr} + \frac{(e^{-hN}W_N)^{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor!}\sum_{r=0}^{N-1} e^{-hr}\right) + \\
& + 2e^{(\alpha-r)\tau}S_0 GN\max\{W_N^2, 1\}\left(e^{2h(\bar{k}+1)}\frac{(e^{-hN}W_N)^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor!}\sum_{r=0}^{N-1} e^{-hr} + e^{-2h(\bar{l}+1)}\frac{(e^{hs}W_N)^{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor!}\sum_{r=0}^{N-1} e^{hr}\right)
\end{aligned}
$$

for $\bar{k}, \bar{l} \geq N\lceil 2e^{hN}W_N - 1\rceil - 1$. Since we also have $hs \leq hN\left\lfloor \frac{s}{N} \right\rfloor + hN$ and $-hs \leq -hN\left\lfloor \frac{s}{N} \right\rfloor$, we can write:

$$
\begin{aligned}
V - V^{TT} \leq & 2e^{(\alpha-r)\tau}S_0 G\left[\frac{(e^{hN}W_N)^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor!}\sum_{r=0}^{N-1} e^{hr} + \frac{(e^{-hN}W_N)^{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor!}\sum_{r=0}^{N-1} e^{-hr}\right. \\
& \left. + N\max\{W_N^2, 1\}\left(e^{2hN}\frac{(e^{hN}W_N)^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor!}\sum_{r=0}^{N-1} e^{-hr} + \frac{(e^{-hN}W_N)^{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor!}\sum_{r=0}^{N-1} e^{hr}\right)\right] \\
\leq & 2e^{(\alpha-r)\tau}S_0 G\left[\frac{(e^{hN}W_N)^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor!}\left(\sum_{r=0}^{N-1} e^{hr} + N\max\{W_N^2, 1\}e^{2hN}\sum_{r=0}^{N-1} e^{-hr}\right) + \frac{(e^{-hN}W_N)^{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor!}\left(\sum_{r=0}^{N-1} e^{-hr} + N\max\{W_N^2, 1\}\sum_{r=0}^{N-1} e^{hr}\right)\right]
\end{aligned}
$$

In order to have the desired inequality, $V - V^{TT} < \varepsilon$, we ask:

$$
\begin{aligned}
\frac{(e^{hN}W_N)^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor!}\left(\sum_{r=0}^{N-1} e^{hr} + N\max\{W_N^2, 1\}e^{2hN}\sum_{r=0}^{N-1} e^{-hr}\right) &< \frac{\varepsilon}{4e^{(\alpha-r)\tau}S_0 G} \\
\frac{(e^{-hN}W_N)^{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor!}\left(\sum_{r=0}^{N-1} e^{-hr} + N\max\{W_N^2, 1\}\sum_{r=0}^{N-1} e^{hr}\right) &< \frac{\varepsilon}{4e^{(\alpha-r)\tau}S_0 G}.
\end{aligned}
$$

Let us call

$$k_+ = \sum_{r=0}^{N-1} e^{hr} + N\max\{W_N^2, 1\}e^{2hN}\sum_{r=0}^{N-1} e^{-hr}$$
$$k_- = \sum_{r=0}^{N-1} e^{-hr} + N\max\{W_N^2, 1\}\sum_{r=0}^{N-1} e^{hr}.$$

Using Lemma 5.3 we impose:

$$e^{hN+1}W_N - \left\lfloor \frac{\bar{k}+1}{N} \right\rfloor \leq \ln\varepsilon - \ln(4S_0G) - (\alpha - r)\tau - \ln k_+$$

$$e^{-hN+1}W_N - \left\lfloor \frac{\bar{l}+1}{N} \right\rfloor \leq \ln\varepsilon - \ln(4S_0G) - (\alpha - r)\tau - \ln k_-$$

which means

$$\bar{k} \geq N\left\lceil e^{hN+1}W_N - \ln\varepsilon + \ln(4S_0G) + (\alpha - r)\tau + \ln k_+ \right\rceil - 1$$

$$\bar{l} \geq N\left\lceil e^{-hN+1}W_N - \ln\varepsilon + \ln(4S_0G) + (\alpha - r)\tau + \ln k_- \right\rceil - 1$$

Adding the conditions for Proposition 5.11, we have:

$$\bar{k} \geq \max\{N\left\lceil e^{hN+1}W_N - \ln\varepsilon + \ln(4S_0G) + (\alpha - r)\tau + \ln k_+ \right\rceil - 1, N\left\lceil 2e^{hN}W_N - 1 \right\rceil - 1\} \tag{10}$$
$$\bar{l} \geq \max\{N\left\lceil e^{-hN+1}W_N - \ln\varepsilon + \ln(4S_0G) + (\alpha - r)\tau + \ln k_- \right\rceil - 1, N\left\lceil 2e^{hN}W_N - 1 \right\rceil - 1\} \tag{11}$$

◇

4. Proof of Theorem 4.2

Theorem: *Given $\varepsilon > 0$, considering V the HS European put option value, taking $\bar{k} \geq \max\{N\lceil 2W_N - 1\rceil - 1, N\lceil W_N e - \ln\varepsilon - r\tau + \ln(4N(N+1)KG)\rceil - 1\}$, we have that the European put option value V^{TT} obtained via truncation of the tree at levels $\bar{k}$ and $-\bar{l}$ with $\bar{l} = \bar{k}$ satisfies*

$$V - V^{TT} < \varepsilon.$$

Proof:

Taking $\bar{l} = \bar{k}$ in Equation (5.34) in the main article, we have

$$V - V^{TT} \leq 2e^{-r\tau} K(N+1) \sum_{k=\bar{k}+1}^{Nn} \widetilde{Q}_N(k) \tag{12}$$

Applying Proposition 5.11 to Equation (12) we obtain:

$$V - V^{TT} \leq 4e^{-r\tau} K(N+1) GN \frac{W_N^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor !}$$

for $\bar{k} \geq N\lceil 2W_N - 1 \rceil - 1$.

In order for it to be less than an arbitrary ε, we impose $\bar{k} \geq N\lceil W_N e - \ln \varepsilon - r\tau + \ln(4N(N+1)KG) \rceil - 1$.

Collecting all requirements on $\bar{k}$, we get

$\bar{k} \geq \max\{N\lceil 2W_N - 1 \rceil - 1, N\lceil W_N e - \ln \varepsilon - r\tau + \ln(4N(N+1)KG) \rceil - 1\}$.

$\diamond$

5. Proof of Lemma 6.1

Lemma: $V_E^0(0,0,0) = V^{TT}$.

Proof: We want to show that the value V^{TT} coincides with the value $V_E^0(0,0,0)$ obtained via backward procedure according to the following formula: $V_E^0(i,j,k) = e^{-r\Delta t} \sum_{l=-N}^{N} (V_E^0(i+1, j+1, k+l)p + V_E^0(i+1, j, k+l)(1-p))q_l$ if $k \in [-\bar{l}, \bar{k}]$, 0 otherwise; with initial data $V_E^0(n,j,k) = 0$ for j integer between 0 and n and k integer such that $-nN \leq k \leq -\bar{l} - 1$ or $\bar{k} + 1 \leq k \leq nN$, and $V_E^0(n,j,k) = (S(n,j,k) - K)^+$ for the call option, $V_E^0(n,j,k) = (K - S(n,j,k))^+$ for the put option, for j integer between 0 and n and k integer such that $-\bar{l} \leq k \leq \bar{k}$.

Let us denote as B the class of all paths on the tree that go from the node $(0,0,0)$ to one of the nodes (n,j,k) at maturity τ. For any $\beta \in B$ we will denote by prob(β) the probability of following β and value(β) the value of the option at the end of the path β. Let us denote $B_{[-\bar{l},\bar{k}]}$ the class of all the paths on the tree that go from the node $(0,0,0)$ to one of the nodes at maturity without trespassing the $-\bar{l}$ and $\bar{k}$ boundaries, that is, where every node (i,j,k) of the path has $-\bar{l} \leq k \leq \bar{k}$.

The expression

$$e^{-r\tau} \sum_{\beta \in B_{[-\bar{l},\bar{k}]}} \text{prob}(\beta) \cdot \text{value}(\beta) \tag{13}$$

coincides with V^{TT}, since they identify the same sum: every path that does not go out of the borders needs to end at a level $-\bar{l} \leq k \leq \bar{k}$; all the paths ending in a node (n, j, k) share the same value for the option, so if we collect all the addenda in (13) that end in the same node we obtain $(K - S_0 e^{(-n+2j)\sigma\sqrt{\Delta t}+hk})^+ P(j) Q_N^T(k)$ in the put case and $(S_0 e^{(-n+2j)\sigma\sqrt{\Delta t}+hk} - K)^+ P(j) Q_N^T(k)$ in the call case.

We will show that the V^{TT} as in (13) coincides with $V_E^0(0,0,0)$ for induction on the number of steps n.

Let us start with $n = 1$. Our tree has only one step, which means that the values at maturity of the option are given by the $2(2N+1)$ children of $(0,0,0)$. In this case $\Delta t = \tau$. Let $0 \leq \bar{l}, \bar{k} \leq N$, that means that $(0,0,0)$ is surely in the allowed zone, while some of its children may not. Since the value of the option on the nodes $(1, j, k)$ with $k \notin [-\bar{l}, \bar{k}]$ is 0, we can write:

$$\begin{aligned} V_E^0(0,0,0) &= e^{-r\tau} \sum_{l=-N}^{N} (V_E^0(1, j+1, l)p + V_E^0(1, j, l)(1-p))q_l = \\ &= e^{-r\tau} \sum_{l=-\bar{l}}^{\bar{k}} V_E^0(1, j+1, l)pq_l + V_E^0(1, j, l)(1-p)q_l = \\ &= e^{-r\tau} \sum_{\beta \in B_{[-\bar{l},\bar{k}]}} \text{prob}(\beta) \cdot \text{value}(\beta) = V^{TT} \end{aligned}$$

where the last equality is due to the fact that in a single step the paths that trespass are those that end outside the boundary.

Let us now suppose the thesis is true for all trees with $n-1$ steps. Let us consider a tree of n steps. In this case $\Delta t = \tau/n$. We focus on the first step and compute the value of the option in $(0,0,0)$, with the backward procedure: $V_E^0(0,0,0) = e^{-r\Delta t} \sum_{l=-N}^{N} (V_E^0(1,1,l)p + V_E^0(1,0,l)(1-p))q_l$.

If $l \notin [-\bar{l}, \bar{k}]$, $V_E^0(1,1,l) = V_E^0(1,0,l) = 0$. Otherwise, we consider the $n-1$ trees that start at $(1, j, l)$ with $j = 0, 1$ and $l \in [-\bar{l}, \bar{k}]$ and end at τ. For such j, l, let us denote $B_{[-\bar{l},\bar{k}]}^{(1,j,l)}$ the class of all the paths on the tree that go from the node $(1, j, l)$ to one of the nodes (n, j, k) at maturity without going out of the $[-\bar{l}, \bar{k}]$ zone. On these smaller trees we apply induction and write the values $V_E^0(1, j, l)$ as

$$V_E^0(1, j, l) = e^{-r\tau'} \sum_{\beta' \in B_{[-\bar{l},\bar{k}]}^{(1,j,l)}} \text{prob}(\beta') \cdot \text{value}(\beta')$$

where we indicated with τ' the time interval $\tau' = \Delta t(n-1)$.

Therefore we can write

$$\begin{aligned} V_E^0(0,0,0) &= e^{-r\Delta t} \sum_{\substack{l=-N \\ l\in[-\bar{l},\bar{k}]}}^{N} (V_E^0(1,1,l)p + V_E^0(1,0,l)(1-p))q_l \\ &= e^{-r\tau} \sum_{\substack{l=-N \\ l\in[-\bar{l},\bar{k}]}}^{N} \left(\sum_{\beta'\in B^{(1,1,l)}_{[-\bar{l},\bar{k}]}} \text{prob}(\beta')\cdot\text{value}(\beta')pq_l + \sum_{\beta'\in B^{(1,0,l)}_{[-\bar{l},\bar{k}]}} \text{prob}(\beta')\cdot\text{value}(\beta')(1-p)q_l \right) = \\ &= e^{-r\tau} \sum_{\beta\in B_{[-\bar{l},\bar{k}]}} \text{prob}(\beta)\cdot\text{value}(\beta) \end{aligned}$$

where we used the fact that $\Delta t + \tau' = \tau$, and we considered that if a path β that connects the node $(0,0,0)$ to a node at maturity τ (without trespassing) visits node $(1,0,l)$ and is afterwards identical to β', we will have value(β) =value(β') and prob(β) $= (1-p)q_l\cdot$ prob(β'), while if a path β that connects the node $(0,0,0)$ to a node at maturity τ (without trespassing) visits node $(1,1,l)$ and is afterwards identical to β', we will have value(β) = value(β') and prob(β) = $pq_l\cdot$ prob(β').

$\diamond$

6. Proof of Lemma 6.2

Lemma: $V_E^b(0,0,0) = \widehat{V^b}$.

Proof: The proof, similar to that of Lemma 6.1, is written for induction on the number of steps n.

In this situation, in order to understand the contribution of every path to the value of the option, we are interested in when a path, going from $(0,0,0)$ to maturity, first crosses the boundaries. Given any $\beta \in B \setminus B_{[-\bar{l},\bar{k}]}$, we will denote with i($\beta$) the time index $0 \le i \le n$ of the first exit of β from the allowed zone $[-\bar{l},\bar{k}]$.

When $n = 1$, the tree has only one step, which means that the values at maturity of the option are given by the $2(2N+1)$ children of $(0,0,0)$. In this case $\Delta t = \tau$. Let $0 \le \bar{l}, \bar{k} \le N$, that means that $(0,0,0)$ is surely in the allowed zone, while some of its children may be not. Since the value of the option is b on the nodes $(1,j,k)$ with $k \notin [-\bar{l},\bar{k}]$, we can write:

$$
\begin{aligned}
V_E^b(0,0,0) =& e^{-r\tau}\sum_{l=-N}^{N}(V_E^b(1,j+1,l)p + V_E^b(1,j,l)(1-p))q_l = \\
=& e^{-r\tau}\sum_{l=-\bar{l}}^{\bar{k}}(V_E^b(1,j+1,l)pq_l + V_E^b(1,j,l)(1-p)q_l) + e^{-r\tau}\sum_{l=-N}^{-\bar{l}-1} b + e^{-r\tau}\sum_{l=\bar{k}+1}^{N} b = \\
=& e^{-r\tau}\sum_{\beta\in B_{[-\bar{l},\bar{k}]}} \text{prob}(\beta)\cdot\text{value}(\beta) + \sum_{\beta\in B\setminus B_{[-\bar{l},\bar{k}]}} \text{prob}(\beta)\cdot be^{-r\Delta t\,\mathrm{i}(\beta)} \\
=& V^{TT} + \sum_{\beta\in B\setminus B_{[-\bar{l},\bar{k}]}} \text{prob}(\beta)\cdot be^{-r\Delta t\,\mathrm{i}(\beta)} = \widehat{V^b}
\end{aligned}
$$

where we take into account the fact that in a single step the paths that trespass are those that end outside the boundaries.

Let us now suppose the thesis is true for all trees with $n-1$ steps. Let us consider a tree of n steps. In this case $\Delta t = \tau/n$. We focus on the first step and compute the value of $V_E^b(0,0,0)$ with the backward procedure: $V_E^b(0,0,0) = e^{-r\Delta t}\sum_{l=-N}^{N}(V_E^b(1,1,l)p + V_E^b(1,0,l)(1-p))q_l$.

If $l \notin [-\bar{l},\bar{k}]$, $V_E^b(1,1,l) = V_E^b(1,0,l) = b$.

$$
V_E^b(0,0,0) = e^{-r\Delta t}\sum_{\substack{l=-N\\ l\in[-\bar{l},\bar{k}]}}^{N}(V_E^b(1,1,l)p + V_E^b(1,0,l)(1-p))q_l + e^{-r\Delta t}\sum_{\substack{l=-N\\ l\notin[-\bar{l},\bar{k}]}}^{N} bq_l
$$

If $l \in [-\bar{l},\bar{k}]$, we can consider the $n-1$ trees that start at $(1,j,l)$ for $j = 0,1$ and end at maturity τ. For any such j,l, we will denote as $B^{(1,j,l)}$ the class of all paths starting from $(1,j,l)$ and ending at maturity. For any $\beta' \in B^{(1,j,l)} \setminus B^{(1,j,l)}_{[-\bar{l},\bar{k}]}$, $\mathrm{i}(\beta')$ is the time index $0 \le i \le n$ of the first exit of β' from the allowed zone $[-\bar{l},\bar{k}]$.

We apply induction and write that the value $V_E^b(1,j,l)$ for this smaller trees is given by

$$
V_E^b(1,j,l) = e^{-r\tau'}\sum_{\beta'\in B^{(1,j,l)}_{[-\bar{l},\bar{k}]}} \text{prob}(\beta')\cdot\text{value}(\beta') + \sum_{\beta'\in B^{(1,j,l)}\setminus B^{(1,j,l)}_{[-\bar{l},\bar{k}]}} \text{prob}(\beta')\cdot be^{-r\Delta t\,\mathrm{i}(\beta')}
$$

where τ' indicates $\tau' = \tau - \Delta t$, $\Delta t' = \tau'/(n-1)$.

Therefore

$$V_E^b(1,j,l) = e^{-r\tau}\sum_{l=-N}^{N}\left(pq_l \sum_{\beta'\in B^{(1,1,l)}_{[-\bar{l},\bar{k}]}} \text{prob}(\beta')\cdot\text{value}(\beta') + pq_l \sum_{\beta'\in B^{(1,1,l)}\setminus B^{(1,1,l)}_{[-\bar{l},\bar{k}]}} \text{prob}(\beta')\cdot be^{-r\Delta t \text{i}(\beta')} + \right.$$

$$\left. +(1-p)q_l \sum_{\beta'\in B^{(1,0,l)}_{[-\bar{l},\bar{k}]}} \text{prob}(\beta')\cdot\text{value}(\beta') + (1-p)q_l \sum_{\beta'\in B^{(1,0,l)}\setminus B^{(1,j,l)}_{[-\bar{l},\bar{k}]}} \text{prob}(\beta')\cdot be^{-r\Delta t \text{i}(\beta')}\right) +$$

$$+ e^{-r\Delta t}\sum_{\substack{l=-N \\ l\notin[-\bar{l},\bar{k}]}}^{N} bq_l$$

Applying Lemma 6.1, we can rewrite the previous expression introducing the values $V_E^0(1,j,l)$.

$$V_E^b(0,0,0) = e^{-r\tau}\sum_{\substack{l=-N \\ l\in[-\bar{l},\bar{k}]}}^{N}\left(pq_l V_E^0(1,j,l) + (1-p)q_l V_E^0(1,0,l) + \right.$$

$$\left. +pq_l \sum_{\beta'\in B^{(1,1,l)}\setminus B^{(1,1,l)}_{[-\bar{l},\bar{k}]}} \text{prob}(\beta')\cdot be^{-r\Delta t \text{i}(\beta')} + (1-p)q_l \sum_{\beta'\in B^{(1,0,l)}\setminus B^{(1,j,l)}_{[-\bar{l},\bar{k}]}} \text{prob}(\beta')\cdot be^{-r\Delta t \text{i}(\beta')}\right) +$$

$$+ e^{-r\Delta t}\sum_{\substack{l=-N \\ l\notin[-\bar{l},\bar{k}]}}^{N} bq_l$$

Now we consider a path β starting from the node $(0,0,0)$, visiting node $(1,j,l)$ and reaching maturity trespassing the boundaries. We call β' the path going from $(1,j,l)$ to maturity which visits the same nodes as β. If $j=0$ then $\text{prob}(\beta) = (1-p)q_l\ \text{prob}(\beta')$, while if $j=1$ $\text{prob}(\beta) = pq_l\ \text{prob}(\beta')$. If $l\notin[-\bar{l},\bar{k}]$, then $\text{i}(\beta)=1$, otherwise $\text{i}(\beta) = \text{i}(\beta')+1$. This means we can write

$$\begin{aligned} V_E^b(0,0,0) &= V_E^0(0,0,0) + \\ &+ \sum_{\substack{\beta\in B\setminus B_{[-\bar{l},\bar{k}]} \\ \text{i}(\beta)>1}} \text{prob}(\beta)\cdot be^{-r\Delta t \text{i}(\beta)} + \\ &+ \sum_{\substack{\beta\in B\setminus B_{[-\bar{l},\bar{k}]} \\ \text{i}(\beta)=1}} \text{prob}(\beta)\cdot be^{-r\Delta t \text{i}(\beta)} = \\ &= \widehat{V^b} \end{aligned}$$

$\diamond$

7. Proof of Lemma 6.3

Lemma: *Given* $\varepsilon > 0$, *taking* $G = 2N \max\{W_N, 1\} \prod_{i=1}^{N-1} M_i^2 e^{W_N}$, *the values* $\widehat{V^K}$ *and* V^{TT} *obtained via truncation of the tree at levels* $\overline{k}$ *and* $-\overline{k}$, *with* $\overline{k}$ *the smallest integer which satisfies:*

$\overline{k} \geq \max\{N\lceil 2W_N - 1\rceil - 1, N\lceil W_N e - \ln\varepsilon + \ln(4N(N+1)KG)\rceil - 1\}$, *we have*

$$\left|\widehat{V^K} - V^{TT}\right| < \varepsilon$$

Proof:

$$\widehat{V^K} - V^{TT} = \sum_{\beta\in B\setminus B_{[-\bar{l},\bar{k}]}} \text{prob}(\beta) \cdot K e^{-r\Delta t \mathbf{i}(\beta)}$$

For brevity, let us call B^k the set of all paths in $B \setminus B_{[-\bar{l},\bar{k}]}$ which reach a node (n, j, k), with $0 \leq j \leq n$, at maturity. We have:

$$\begin{aligned}
\widehat{V^K} - V^{TT} \leq & K \sum_{k=-Nn}^{Nn} \sum_{\beta\in B^k} \text{prob}(\beta) \\
\leq & K \sum_{k=-Nn}^{-\bar{l}-1} \sum_{\beta\in B^k} \text{prob}(\beta) + K \sum_{k=-\bar{l}}^{\bar{k}} \sum_{\beta\in B^k} \text{prob}(\beta) + K \sum_{k=\bar{k}+1}^{Nn} \sum_{\beta\in B^k} \text{prob}(\beta) \\
\leq & K \sum_{k=-Nn}^{-\bar{l}-1} Q_N(k) + K \sum_{k=\bar{k}+1}^{Nn} Q_N(k) + \\
& + K \sum_{k=-\bar{l}}^{\bar{k}} \sum_{\substack{\beta\in B^k \\ \text{first trespassing } -\bar{l}}} \text{prob}(\beta) + K \sum_{k=-\bar{l}}^{\bar{k}} \sum_{\substack{\beta\in B^k \\ \text{first trespassing } \bar{k}}} \text{prob}(\beta) \\
\leq & K \sum_{k=\bar{l}+1}^{Nn} \widetilde{Q}_N(k) + K \sum_{k=\bar{k}+1}^{Nn} \widetilde{Q}_N(k) + K \sum_{k=-\bar{l}}^{\bar{k}} Q_{\bar{l}}(k) + K \sum_{k=-\bar{l}}^{\bar{k}} Q^{\bar{k}}(k).
\end{aligned}$$

Therefore we have

$$
\begin{aligned}
\widehat{V^K} - V^{TT} &\leq K(N+1)\left(\sum_{k=\bar{l}+1}^{Nn} \widetilde{Q}_N(k) + \sum_{k=\bar{k}+1}^{Nn} \widetilde{Q}_N(k)\right) \\
&\leq 2K(N+1)\sum_{k=\bar{k}+1}^{Nn} \widetilde{Q}_N(k) \\
&\leq 4K(N+1)GN\frac{W_N^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor !}
\end{aligned}
$$

for $\bar{l} = \bar{k} \geq N\lceil 2W_N - 1\rceil - 1$ and applying Equation (2).

We ask $\bar{k} \geq N\lceil W_N e - \ln \varepsilon + \ln(4N(N+1)KG)\rceil - 1$, in order to have

$$
4e^{-r\tau}K_0(N+1)GN\frac{W_N^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor !} < \varepsilon.
$$

Collecting all the requirements on $\bar{k}$, we get that for

$\bar{k} \geq \max\{N\lceil 2W_N - 1\rceil - 1, N\lceil W_N e - \ln \varepsilon + \ln(4N(N+1)KG)\rceil - 1\}$

we have

$$
\left|\widehat{V^K} - V^{TT}\right| < \varepsilon.
$$

◇