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Linearization of a free boundary problem in corrosion detection

Elio Cabib[*], Dario Fasino[†], Eva Sincich[‡]

Abstract

We consider a boundary identification problem arising in nondestructive testing of materials. The problem is to recover a part $\Gamma_I \subset \partial\Omega$ of the boundary of a bounded, planar domain Ω from one Cauchy data pair of a harmonic potential u in Ω collected on a different boundary subset $\Gamma_A \subset \partial\Omega$. We prove Fréchet differentiability of a suitably defined forward map, and discuss local uniqueness and Lipschitz stability results for the linearized problem.

Mathematics Subject Classification: 35R30, 35R25, 31B20.

1 Introduction

In this paper we discuss an inverse problem arising in the nondestructive testing of materials [7, 9, 16]. Such materials are typically metallic specimens, as for instance pipes transporting water, gas, chemically aggressive fluids or bodywork of aircraft, cars, etc., whose surfaces have been damaged by a corrosion attack. In practice, it often happens that such surfaces are not accessible to direct inspection, hence in order to detect the possible presence of corrosion one has to rely on measurements only performed on the accessible part of the specimen surface. In what follows, we assume that a stationary (thermic or electric) potential u is available from direct measurements on the accessible boundary; for definiteness, we will refer generally to u as an electric potential. From these considerations one obtains an inverse problem for the following elliptic boundary value problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \phi & \text{on } \Gamma_A \\ \dfrac{\partial u}{\partial \nu} + \gamma u = 0 & \text{on } \Gamma_I \\ u = 0 & \text{on } \Gamma_D. \end{cases} \tag{1.1}$$

According to this model, Ω represents a conductor which contains no sources and no sinks, so that the potential u is harmonic. We assume that the boundary $\partial\Omega$ is decomposed in three open and disjoint subsets $\Gamma_A, \Gamma_I, \Gamma_D$. On the portion Γ_A, which is the one accessible to direct inspection, we prescribe a current density ϕ and we measure the corresponding voltage potential $u|_{\Gamma_A}$. The portion Γ_I, where corrosion took place, is out of reach. On such a portion, the potential u satisfies a Robin type condition, which models a resistive coupling with the exterior environment, where the Robin coefficient $\gamma > 0$ models an impedance. The remaining portion of the boundary Γ_D is assumed to be grounded.

In this paper we are interested in the inverse problem of determining the location of the unknown and corroded boundary Γ_I from the data collected on the accessible part of the boundary Γ_A, that is, the Cauchy data pair $(\phi, u|_{\Gamma_A})$. In particular, we generalize the main results in [9, 16] to much more general domains.

Many authors have treated analogous boundary identification problems where an unknown boundary is endowed by Neumann or Dirichlet boundary conditions, see for instance [2, 6, 13, 14, 15, 17]. Moreover,

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inverse problems concerning the identification of the impedance coefficient γ in (1.1) (or variants of it) have been addresses e.g., in [3, 5, 7, 9, 13]. For what concerns the determination of a portion of the boundary, where a Robin type condition is prescribed, we recall that in [7] it is proved, by counterexamples, that a single measurement is not sufficient to determine simultaneously the shape of Γ_I and the impedance coefficient γ, and the same holds if the only aim is to determine Γ_I and γ is a fixed constant.
However, we observe that the negative results in [7] concern domains whose unknown boundary Γ_I contain corners. Actually, a convergent numerical scheme for the reconstruction of Γ_I (with a known constant γ) is also shown in [7], under the assumption that Γ_I can be parametrized by a smooth function.
Furthermore, in [5] it has been achieved a global uniqueness result for the simultanous determinantion of Γ_I and γ by means of two measurements, one of which is given for a positive current ϕ, and in [9, 16] the authors prove that, in a rather particular setting with a rectangular domain, one suitable data set collected in the accessible boundary identifies θ uniquely. Moreover, in [18] a local uniqueness result and two reconstruction algorithms by two suitably chosen measurements are presented.
In the present paper we assume that the Robin coefficient γ is known and constant. Moreover, in order to have a solution u to (1.1) with constant sign, we will consider only positive fluxes ϕ, which is also in accordance with the hypothesis required in [5, 9, 18]. In order to investigate the location of the supposed damage, we adopt a model in which the undamaged domain $\Omega \subset \mathbb{R}^2$ is modified by a corrosion process localized on Γ_I. Since we are assuming that external physical conditions do not change significantly, we will consider small perturbations of Γ_I and we analyze the problem by a local approach.
We describe such a situation by introducing a small vector field $\theta \in C_0^1(\Gamma_I)$ so that the damaged domain Ω_θ is such that

$$\partial\Omega_\theta = \overline{\Gamma_A} \cup \overline{\Gamma_D} \cup \overline{\Gamma_{I,\theta}}$$

where $\Gamma_{I,\theta} = \{z \in \mathbb{R}^2 : z = w + \theta(w),\ w \in \Gamma_I\}$. Hence our inverse problem may be reformulated as follows: Find $\theta \in C_0^1(\Gamma_I)$ given a single measurement $(\phi, u|_{\Gamma_A})$ with $\phi \geqslant 0$.
In Section 2 we collect technical details required in the rest of the paper. In Section 3 we consider the forward map

$$F : \theta \mapsto u|_{\Gamma_A}$$

and, by adapting the techniques developed in [11, 12], we show that F is Fréchet differentiable at Γ_I.
In Section 4 we study the linearized problem and discuss some stability properties, provided that

$$2H(x) + \gamma > 0 \tag{1.2}$$

where $H(x)$ is the mean curvature of the undamaged boundary Γ_I. Let us observe that the hypothesis (1.2) is well justified when, for instance, Ω models a $2D$ transverse section of a metallic plate being, in that case, the curvature equal to zero. In particular, we prove a local uniqueness result for θ and a Lipschitz stability result 'à la Bellout and Friedman' [4], by establishing that the Gâteaux derivative does not vanish. Furthermore, we give a quantitative bound of the L^1-norm of θ in an inner portion of Γ_I in terms of the solution u' to the linearized problem on Γ_A. Finally, we observe that the Fréchet differential operator is compact over suitable spaces, hence the local identification issue of θ may be reformulated as the regularized inversion of a compact operator.

2 Definitions and assumptions

Throughout this paper, let Ω be a bounded domain in $\mathbb{R}^2$. We will refer to Ω as the *undamaged* or *reference* domain. As already stated in the Introduction, we consider an inverse problem for the elliptic equation (1.1). Recall that γ is assumed to be a known positive constant.
We denote by B_r the ball in $\mathbb{R}^2$ centerd in zero with radius r. We borrow from [10] the two following definitions:

Definition 2.1. *We shall say that the boundary $\partial\Omega$ of Ω is of Lipschitz class with constants r_0, $M > 0$ if for every $P \in \partial\Omega$ there exists a rigid transformation of coordinates under which we have $P = 0$ and*

$$\Omega \cap B_{r_0} = \{(x, y) : y > g(x)\},$$

where

$$g:(-r_0,r_0)\subset\mathbb{R}\to\mathbb{R}$$

fulfills $g(0)=0$ *and*

$$\|g\|_{C^{0,1}((-r_0,r_0))}\leqslant Mr_0,$$

with the notation

$$\|g\|_{C^{0,1}((-r_0,r_0))}=\|g\|_{L^\infty((-r_0,r_0))}+\ r_0\sup_{\substack{x_1,x_2\in(-r_0,r_0)\\ x_1\neq x_2}}\frac{|g(x_1)-g(x_2)|}{|x_1-x_2|}.$$

Definition 2.2. *Given an integer* $k\geqslant 1$ *and a scalar* $\alpha,\ 0<\alpha<1$, *we shall say that a portion* S *of* $\partial\Omega$ *is of* class $C^{k,\alpha}$ with constants $r_0,\ M>0$ *if for any* $P\in S$ *there exists a rigid transformation of coordinates under which we have* $P=0$ *and*

$$\Omega\cap B_{r_0}=\{(x,y):y>\varphi(x)\}$$

where

$$\varphi:(-r_0,r_0)\subset\mathbb{R}\to\mathbb{R}$$

is a $C^{k,\alpha}$ *function satisfying* $|D^\ell\varphi(0)|=0$ *for* $0\leqslant\ell\leqslant k$ *and*

$$\|\varphi\|_{C^{k,\alpha}((-r_0,r_0))}\leqslant Mr_0,$$

where we denote

$$\begin{aligned}\|\varphi\|_{C^{k,\alpha}((-r_0,r_0))} \ =\ & \sum_{j=0}^{k}\|D^j\varphi\|_{L^\infty((-r_0,r_0))}+\\ &+{r_0}^{k+\alpha}\sup_{\substack{x_1,x_2\in(-r_0,r_0)\\ x_1\neq x_2}}\frac{|D^k\varphi(x_1)-D^k\varphi(x_2)|}{|x_1-x_2|^\alpha}.\end{aligned}$$

Hereafter, we list assumptions and a-priori informations that will hold true throughout this paper.

- **Assumptions on the domain:** Recall that Ω is a bounded domain in $\mathbb{R}^2$. We suppose that there exist constants $r_0,\ M>0$, and $0<\alpha<1$, such that $\partial\Omega$ is of Lipschitz class with constants r_0,M, see Definition 2.1, and that the portion of the boundary Γ_I is of class $C^{2,\alpha}$ with constants r_0,M, see Definition 2.2.

- **Assumptions on the prescribed current density:** We assume that the flux ϕ is such that

  $$\|\phi\|_{H^{-\frac{1}{2}}(\Gamma_A)}\leqslant G$$

  for some positive constant G.

- **Assumptions on u:** We assume that there exists a constant $U>0$ such that

  $$\|u\|_{C^2(\Gamma_I)}\leqslant U\ . \tag{2.1}$$

  We observe that, based on the aforementioned assumptions on Ω and ϕ, the assumption (2.1) can be fulfilled by limiting ourselves to particular geometries, as for instance a cylinder, or by supposing that Γ_I is a connected component of the boundary $\partial\Omega$, see [10].

- **A priori informations on θ:** We suppose that θ is a vector in $C_0^1(\Gamma_I)$ such that if

  $$\theta_\nu(x)=\theta(x)\cdot\nu(x)\equiv 0\ ,\quad x\in\Gamma_I$$

  where $\nu(x)$ is the unit outward normal in $x\in\Gamma_I$, then

  $$\theta(x)\equiv 0\quad x\in\Gamma_I.$$

Moreover, denoting with φ the acute angle such that

$$|\theta_\nu| = |\theta||\cos(\varphi)|$$

we assume that

$$|\cos(\varphi)| \geqslant A > 0. \tag{2.2}$$

In what follows, the constants $r_0, M, \alpha, G, \gamma, U, A$ will be referred to as the *a priori data.*
Any sufficiently small vector field $\theta : \Gamma_I \mapsto \mathbb{R}^2$, $\theta \in C_0^1(\Gamma_I)$ induces a perturbation of $\partial\Omega$ which is still the boundary of a domain that we denote with Ω_θ, with

$$\partial\Omega_\theta = \overline{\Gamma_A} \cup \overline{\Gamma_D} \cup \overline{\Gamma_{I,\theta}}$$

where

$$\Gamma_{I,\theta} = \{z \in \mathbb{R}^2 : z = w + \theta(w),\ w \in \Gamma_I\}.$$

For notational convenience, we will generally identify $\Gamma_{I,\theta}$ with the vector field θ defining it; in particular, the reference boundary Γ_I corresponds to $\theta = 0$. Furthermore, we denote by ν the outward normal to the boundary Γ_I, and by θ_ν and θ_t the normal and tangential components of the field θ, respectively.

Definition 2.3. *We shall denote with F the* forward map

$$\begin{array}{rccc} F: & C_0^1(\Gamma_I) & \rightarrow & H^{\frac{1}{2}}(\Gamma_A) \\ & \theta & \mapsto & u_\theta|_{\Gamma_A} \end{array} \tag{2.3}$$

where $u_\theta \in H_0^1(\Omega, \Gamma_D)$ is the solution to the problem

$$\begin{cases} \Delta u_\theta = 0 & \textit{in } \Omega_\theta \\ \dfrac{\partial u_\theta}{\partial \nu} = \phi & \textit{on } \Gamma_A \\ \dfrac{\partial u_\theta}{\partial \nu} + \gamma u_\theta = 0 & \textit{on } \Gamma_{I,\theta} \\ u = 0 & \textit{on } \Gamma_D. \end{cases} \tag{2.4}$$

With the help of the foregoing definition, we can state our boundary identification problem as the solution on the nonlinear equation $F(\theta) = \eta$ for a given $\eta = u|_{\Gamma_A}$, the trace on the accessible boundary Γ_A of the potential u that solves (1.1) with a prescribed flux ϕ.

3 Fréchet differentiability of the forward map

This section contains the main results of this paper. In the forthcoming theorem, we prove that the forward map introduced in Definition 2.3 is Fréchet differentiable (for $\theta = 0$), and provide the explicit form of the derivative. In the subsequent corollary, we specialize this result to the case where Ω is a rectangle, as considered in [9].

Theorem 3.1. *The operator F in (2.3) is Fréchet differentiable at Γ_I, namely*

$$\frac{1}{\|\theta\|_{C_0^1(\Gamma_I)}} \|F(\Gamma_{I,\theta}) - F(\Gamma_I) - F'(\Gamma_I)\theta\|_{H^{\frac{1}{2}}(\Gamma_A)} \to 0 \qquad \textit{as } \theta \to 0$$

with derivative $F'(\Gamma_I)\theta = u'|_{\Gamma_A}$, where u' is the solution to the following boundary value problem

$$\begin{cases} \Delta u' = 0 & \textit{in } \Omega \\ \dfrac{\partial u'}{\partial \nu} = 0 & \textit{on } \Gamma_A \\ \dfrac{\partial u'}{\partial \nu} + \gamma u' = \dfrac{\mathrm{d}}{\mathrm{d}s}\left(\theta_\nu \dfrac{\mathrm{d}}{\mathrm{d}s} u\right) + \gamma\theta_\nu\left(\gamma + 2H\right) u & \textit{on } \Gamma_I \\ u' = 0 & \textit{on } \Gamma_D, \end{cases} \tag{3.1}$$

where H denotes the mean curvature of the boundary Γ_I.

Proof. Let us recall that a weak solution to the problem (3.1) is a function $u' \in H^1_0(\Omega, \Gamma_D)$ such that

$$\int_\Omega \nabla u' \nabla v + \int_{\Gamma_I} \gamma u' v = \gamma \int_{\Gamma_I} \theta_\nu (\gamma + 2H) uv - \int_{\Gamma_I} \theta_\nu \frac{\mathrm{d}u}{\mathrm{d}s}\frac{\mathrm{d}v}{\mathrm{d}s} \tag{3.2}$$

for all $v \in H^1_0(\Omega, \Gamma_D)$. Moreover, recall that the Sobolev space $H^1_0(\Omega, \Gamma_D)$ is defined as follows:

$$H^1_0(\Omega, \Gamma_D) = \{v \in H^1(\Omega) : \ v = 0 \text{ on } \Gamma_D \text{ in the trace sense}\}.$$

With a little abuse of notation, we denote by $\theta \in C^1(\Omega)$ a smooth prolongation of the original vector field θ to the whole Ω which satisfies $\theta(x) = 0$ on Γ_A and $\|\theta\|_{C^1(\Omega)} \leqslant c\|\theta\|_{C^1_0(\Gamma_I)}$ where $c > 0$ depends on the a priori data only. Note that, with this convention, the theorem can be proved by showing the limit

$$\frac{\|u_\theta - u - u'\|_{H^{\frac{1}{2}}(\Gamma_A)}}{\|\theta\|_{C^1(\Omega)}} \to 0$$

as $\|\theta\|_{C^1(\Omega)} \to 0$, where u_θ is the solution of (2.4). Therefore, we introduce a change of variables defined onto the unperturbed domain Ω:

$$\varphi : \Omega \to \Omega_\theta, \qquad \varphi(x) = x + \theta(x).$$

Then, consider the function $\tilde{u_\theta} = u_\theta \circ \varphi$ and the bilinear form

$$R_\theta(\tilde{u_\theta}, v) := \int_\Omega (\nabla \tilde{u_\theta} J_\psi J_\psi^T \nabla v) \det J_\varphi + \int_{\Gamma_I} \gamma \tilde{u_\theta} v \det \tilde{J}_\varphi$$

for any $v \in H^1(\Omega)$, where J_φ denotes the Jacobian of φ, ϕ the inverse of φ with Jacobian J_ψ and $\tilde{J}_\varphi$ the Jacobian of φ with respect to the surface integral.
Since u and u_θ have the same Neumann data ϕ on Γ_A we conclude that

$$R(u, v) = R_\theta(\tilde{u_\theta}, v) \qquad \forall v \in H^1_0(\Omega, \Gamma_D), \tag{3.3}$$

where

$$R(u, v) = \int_\Omega \nabla u \nabla v + \int_{\Gamma_I} \gamma uv$$

is the bilinear form associated to (3.2).
The a priori regularity assumption (2.1) on u implies that $\frac{d^2u}{d^2s} \in (H^{\frac{1}{2}}_{00}(\Gamma_I))^*$ and hence there exists a unique solution $u' \in H^1_0(\Omega, \Gamma_D)$ to the problem (3.1). We define $w = u' + \theta \cdot \nabla u$ and we notice that $u'|_{\Gamma_A} = w|_{\Gamma_A}$.
By the coercitivity of R it is sufficient to prove that

$$\forall v \in H^1_0(\Omega, \Gamma_D), \qquad \frac{1}{\|\theta\|_{C^1(\Omega)}} R(\tilde{u_\theta} - u - w, v) \to \ 0$$

when θ tends to zero. By (3.3) we obtain that

$$\begin{aligned} R(u - \tilde{u_\theta}, v) &= R_\theta(\tilde{u_\theta}, v) - R(\tilde{u_\theta}, v) \\ &= \int_\Omega \nabla \tilde{u_\theta} (J_\psi J_\psi^T \det J_\varphi - I) \nabla v + \int_{\Gamma_I} \gamma (\det \tilde{J}_\varphi - 1) \tilde{u_\theta} v. \end{aligned} \tag{3.4}$$

Dealing as in [11, Theorem 2.1] and [12] we can infer that

$$\|J_\psi J_\psi^T \det J_\varphi - I + J_\theta + J_\theta^T - \operatorname{div} \theta I\|_{C^0(\Omega)} = \mathcal{O}(\|\theta\|^2_{C^1(\Omega)}), \tag{3.5}$$

$$\|\det \tilde{J}_\varphi - 1 - \operatorname{div} \theta_t + 2H\theta_\nu\|_{C^0(\Gamma_I)} = \mathcal{O}(\|\theta\|^2_{C^1(\Omega)}). \tag{3.6}$$

By the estimates (3.5), (3.6) and by (3.4) we deduce by coercivity that

$$\|\tilde{u_\theta} - u\|_{H^1(\Omega)} \to 0 \ \text{ as } \ \theta \to 0.$$

Therefore, due to the equation (3.4) it remains to prove that

$$R(w,v) = \int_\Omega \nabla u(J_\theta + J_\theta^T - \operatorname{div}\theta I)\nabla v + \int_{\Gamma_I} \gamma u(\operatorname{div}\theta_t - 2H\theta_\nu)v$$

for all $v \in H_0^1(\Omega, \Gamma_D)$.
Since u' is a solution to (3.1) we get from the boundary condition that

$$\begin{aligned} R(w,v) &= \int_\Omega \nabla(\theta\cdot\nabla u)\nabla v - \int_{\Gamma_I}\gamma\left[\theta\cdot\nabla u - \theta_\nu\left(\frac{\partial u}{\partial\nu} - 2Hu\right)\right]v \\ &\quad - \int_{\Gamma_I}\operatorname{div}(\theta_\nu\nabla_t u)v\ . \end{aligned}$$

The formula

$$\operatorname{div}(\nu\times W) = -\nu\cdot\operatorname{curl} W$$

for a vector field $W \in H^1(\Omega)$ yields to

$$\begin{aligned} R(w,v) &= \int_\Omega \nabla(\theta\cdot\nabla u)\nabla v - \int_{\Gamma_I}\gamma\theta_\nu 2Huv + \\ &\quad - \int_{\Gamma_I}\gamma\theta_t\cdot(\nabla_t u)v - \int_{\Gamma_I}\nu\cdot\operatorname{curl}(\theta_\nu(\nabla u\times\nu))v. \end{aligned}$$

The Green's formula for test functions $v \in H^2(\Omega)$ leads to

$$\begin{aligned} R(w,v) &= -\int_\Omega(\theta\cdot\nabla u)\Delta v - \int_{\Gamma_I}(\theta\cdot\nabla u)\frac{\partial v}{\partial\nu} - \int_{\Gamma_I}\gamma\theta_\nu 2Huv + \\ &\quad - \int_{\Gamma_I}\gamma\theta_t\cdot(\nabla_t u)v - \int_{\Gamma_I}\nu\cdot\operatorname{curl}(\theta_\nu(\nabla u\times\nu))v. \end{aligned}$$

Moreover, according to the Green's formula for a vector field $W \in H^2(\Omega)$ and a scalar function $v \in H^1(\Omega)$ we have that

$$\int_{\partial\Omega}\nu\cdot\operatorname{curl}Wv = \int_\Omega \operatorname{curl}W\nabla v = \int_{\partial\Omega}\nu\times W\nabla v. \tag{3.7}$$

Hence, by (3.7) with $W = \theta_\nu(\nabla u\times\nu)$, recalling that $\theta\in C^1(\Omega)$ and by Gauss theorem we deduce that

$$\begin{aligned} R(w,v) &= \int_\Omega \operatorname{div}[(\theta\cdot\nabla u)\nabla v + (\theta\cdot\nabla v)\nabla u - (\nabla u\cdot\nabla v)\theta] - (\theta\cdot\nabla u)\Delta v + \\ &\quad + \int_{\Gamma_I}[(\theta\cdot\nabla v)\nabla u - \theta(\nabla u\cdot\nabla v)]\cdot\nu - \int_{\Gamma_I}\gamma\theta_\nu 2Huv + \\ &\quad - \int_{\Gamma_I}\gamma\theta_t\cdot(\nabla_t u)v + \int_{\Gamma_I}\theta_\nu\nabla_t u\cdot\nabla_t v. \end{aligned}$$

By the formula (see [11])

$$\begin{aligned} \nabla u(J_\theta + J_\theta^T - \operatorname{div}\theta I)\nabla v &= \operatorname{div}[(\theta\cdot\nabla u)\nabla v + (\theta\cdot\nabla v)\nabla u - (\nabla u\cdot\nabla v)\theta] + \\ &\quad -(\theta\cdot\nabla u)\Delta v \end{aligned}$$

we conclude that

$$\begin{aligned} R(w,v) &= \int_\Omega \nabla u(J_\theta + J_\theta^T - \operatorname{div}\theta I)\nabla v - \int_{\Gamma_I}\gamma\theta_\nu 2Huv + \\ &\quad - \int_{\Gamma_I}\gamma\theta_t\cdot(\nabla_t u)v + \int_{\gamma_I}\theta_t\cdot\nabla_t v\frac{\partial u}{\partial\nu}. \end{aligned}$$

From the Robin boundary condition for u and the identity

$$\int_{\Gamma_I} \gamma\theta_t \cdot (\nabla_t uv) = -\int_{\Gamma_I} \gamma uv \operatorname{div}\theta_t$$

for the surface gradient we obtain the thesis. □

Corollary 3.2. *Let $\Omega = (0,a)\times(0,b)$ be such that $\Gamma_A = (0,a)\times\{0\}$, $\Gamma_I = (0,a)\times\{b\}$, $\Gamma_D = \{0\}\times(0,b)\cup\{a\}\times(0,b)$ and let $\theta = (\theta_1,\theta_2)\in C^1_0((0,a))$. Hence $u'\in H^1(\Omega)$ is the solution to*

$$\begin{cases} \Delta u' = 0 & \text{in } \Omega \\ u'(0,y) = u'(a,y) = 0 & y\in(0,b) \\ u'_y(x,b) + \gamma u'(x,b) = \beta(x) & x\in(0,a) \\ u'_y(x,0) = 0 & x\in(0,a) \end{cases}$$

where $\beta(x) = -\theta_2(x)\left(u_{yy}(x,b) - \gamma^2 u(x,b)\right) + \theta_2'(x)u_x(x,b)$.

Proof. The claim follows from Theorem 3.1, noticing that in this special geometry we have $\theta_\nu = \theta_2$, $H = 0$, and the solution u of problem (1.1) is harmonic up to the boundary Γ_I. □

4 Applications

In this section we prove some consequences of our main results in the previous section, which are relevant for the analysis and numerical solution of our boundary identification problem. Indeed, Theorem 4.2 proves that the "domain derivative" operator $F'(\Gamma_I)$ is injective, under some reasonable hypotheses. This fact is relevant to conclude that the solution of our inverse problem is identifiable (i.e., unique whenever it exists), at least for sufficiently small perturbations. Moreover, Theorem 4.3 and Theorem 4.4 give two "local stability" results. In particular, in Theorem 4.4 we prove a lower bound for θ on a suitable portion of Γ_I in terms of $u|_{\Gamma_A} = F'\theta$, thus showing that the inversion of F' is not too much ill-behaved, at least in the conditions stated therein.

From a computational point of view, the availability of the expression of the operator F' allows to tackle the solution of the boundary identification problem by a regularized Newton-type iteration. In this case, the main computational task consists of the solution of a sequence of linear operator equations, associated to the operator F'. The ill-posed character of these linearized problems is clarified by Theorem 4.5.

Hereafter, we denote by Γ_I^ρ a portion of the boundary Γ_I sufficiently distant from its endpoints; more precisely, given $\rho > 0$, we set

$$\Gamma_I^\rho = \{x\in\Gamma_I : \operatorname{dist}(x,\partial\Gamma_I) > \rho\}.$$

Lemma 4.1. *Let $\phi\in L^p(\Gamma_A)$, $p > 2$, be a non-negative a.e. function and let u the solution to the problem (1.1). Then we have that*

$$u(x) > 0 \quad \forall x\in\Gamma_I.$$

Proof. For the proof we refer to [8, Lemma 2]. Moreover, we observe that the arguments in [3, Proposition 2.3] leading to a quantitative control of the vanishing rate of u, in the more difficult case when ϕ has a variable sign, might be adapted in order to achieve the estimate

$$u(x) > c_\rho \quad \forall x\in\Gamma_I^\rho, \tag{4.1}$$

where c_ρ is a positive constant depending on the a-priori data and on ρ only. □

Theorem 4.2 (**Injectivity of F'**)**.** *Let the hypothesis of Lemma 4.1 be satisfied. Let us assume that $2H(x) + \gamma > 0$ and $\theta_\nu(x) \leqslant 0$ for any $x\in\Gamma_I$. Then F' is injective.*

Proof. Let us suppose that $F'\theta = 0$. By Holmgren unique continuation theorem, we have that $u'\equiv 0$ in $\overline{\Omega}$. Hence by (3.1) and by the a-priori regularity assumption (2.1) we have that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\theta_\nu \frac{\mathrm{d}}{\mathrm{d}s} u\right) = -\gamma\theta_\nu(\gamma + 2H)u \quad \text{on } \Gamma_I.$$

By Lemma 4.1 we have that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\theta_\nu \frac{\mathrm{d}}{\mathrm{d}s}u\right) < 0 \quad \text{on } \Gamma_I$$

and thus $\theta_\nu \frac{d}{ds}u$ is decreasing in Γ_I. Since $\theta \in C_0^1(\Gamma_I)$ we have that $\theta_\nu \frac{d}{ds}u = 0$ on $\partial\Gamma_I$ and thus by the monotonicity we infer that $\theta_\nu \frac{d}{ds}u \equiv 0$ in Γ_I. This implies that

$$0 \equiv \frac{d}{ds}\left(\theta_\nu \frac{d}{ds}u\right) = -\gamma\theta_\nu(\gamma + 2H)u \ \text{ on } \Gamma_I.$$

Hence by Lemma 4.1 and by the hypothesis we deduce that $\theta_\nu \equiv 0$ and thus $\theta \equiv 0$. □

Theorem 4.3 (Local Lipschitz stability). *Let the hypothesis of Lemma 4.1 be satisfied and let $2H(x)+\gamma > 0$ for any $x \in \Gamma_I$.*
Given $\bar\theta \in C_0^1(\Gamma_I)$ such that $\bar\theta_\nu = \bar\theta\cdot\nu \leqslant 0$ and given $h \in (h_0, h_0)$ with $h_0 > 0$, we set $\theta_h = h\cdot\bar\theta$. Denoting by u_h the solution to (2.4) with u_θ and $\theta = \theta_h$ we have that

$$\lim_{h\to 0} \frac{\|u_h - u\|_{H^{\frac{1}{2}}(\Gamma_A)}}{|h|} > 0. \tag{4.2}$$

Proof. By Theorem 3.1 we can infer that there exists $\varepsilon(h) \in H_0^1(\Omega)$ such that

$$u_h = u + hv' + \varepsilon(h) \tag{4.3}$$

where $\|\varepsilon(h)\|_{H_0^1(\Omega)} \to 0$ as $h \to 0$ and where $v' \in H_0^1(\Omega)$ is the weak solution to (1.1) with $u' = v'$ and $\theta = \bar\theta$. According to (4.3), we have that (4.2) is equivalent to

$$\|v'\|_{H^{\frac{1}{2}}(\Gamma_A)} > 0\ .$$

Let us assume that $v' = 0$ on Γ_A. Then, arguing as in Theorem 4.2, we will obtain that

$$\gamma\bar\theta_\nu(\gamma + 2H)u \equiv 0 \ \text{ on } \Gamma_I\ .$$

This would imply that u vanishes in a set of positive measure of Γ_I, which is in contradiction with Lemma 4.1. □

Theorem 4.4. *Let the hypothesis of Lemma 4.1 be satisfied. Moreover, let us assume that $2H(x)+\gamma > 0$ and $\theta_\nu \leqslant 0$ for any $x \in \Gamma_I$. Then, for any $\rho > 0$ there exists a positive constant C_ρ depending on the a-priori data and on ρ only such that*

$$\|u'\|_{H^{\frac{1}{2}}(\Gamma_A)} \geqslant C_\rho \int_{\Gamma_I^\rho} |\theta|.$$

Proof. From the weak formulation of the problem (3.1), which is shown in (3.2), we have that

$$\int_\Omega \nabla u'\nabla u = -\int_{\Gamma_I} \gamma u'u + \gamma^2 \int_{\Gamma_I} \theta_\nu u^2 + \gamma \int_{\Gamma_I} \theta_\nu 2Hu^2 + \int_{\Gamma_I} \theta_\nu \left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2.$$

On the other hand, we have that

$$\int_\Omega \nabla u'\nabla u = -\int_{\Gamma_I} \gamma u'u + \int_{\Gamma_A} u'\frac{\partial u}{\partial\nu}.$$

Combining the last two equalities we have that

$$\int_{\Gamma_A} u'\frac{\partial u}{\partial\nu} = \gamma^2 \int_{\Gamma_I} \theta_\nu u^2 + \gamma \int_{\Gamma_I} \theta_\nu 2Hu^2 + \int_{\Gamma_I} \theta_\nu \left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2.$$

By the Schwartz inequality and the hypotheses, we infer that

$$\begin{aligned}
\|u'\|_{H^{\frac{1}{2}}(\Gamma_A)}\|\phi\|_{H^{-\frac{1}{2}}(\Gamma_A)} &\geqslant \left|\int_{\Gamma_I}\gamma\theta_\nu(\gamma+2H)u^2+\int_{\Gamma_I}\theta_\nu\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2\right| \\
&\geqslant \int_{\Gamma_I}|\gamma\theta_\nu(\gamma+2H)|u^2 \\
&\geqslant \int_{\Gamma_I^\rho}|\gamma\theta_\nu(\gamma+2H)|u^2.
\end{aligned}$$

By the estimates (4.1) and (2.2), we infer that there exists a constant $C_\rho>0$ depending on the a priori data and on ρ only such that

$$\|u'\|_{H^{\frac{1}{2}}(\Gamma_A)}\geqslant C_\rho\int_{\Gamma_I^\rho}|\theta|,$$

and the proof is complete. □

Theorem 4.5. *The linear operator*

$$\begin{array}{rccc} F'(\Gamma_I): & C_0^1(\Gamma_I) & \rightarrow & L^2(\Gamma_A) \\ & \theta & \mapsto & u'|_{\Gamma_A} \end{array}$$

is compact.

Proof. Let us first prove that $F'(\Gamma_I)$ as operator from $C_0^1(\Gamma_I)$ to $H^{\frac{1}{2}}(\Gamma_A)$ is bounded. In what follows, we will denote by C a generic positive constant depending on the a-priori data only, whose value may change from one occurrence to another.
By the weak formulation (3.2) with $v=u'$ we have that

$$\int_\Omega|\nabla u'|+\gamma\int_{\Gamma_I}|u'|^2=\gamma\int_{\Gamma_I}\theta_\nu(\gamma+2H)uu'-\int_{\Gamma_I}\theta_\nu\frac{\mathrm{d}u'}{\mathrm{d}s}\frac{\mathrm{d}u}{\mathrm{d}s}.$$

By a Poincaré type inequality we have that there exists a constant C such that

$$\|u'\|^2_{H^1(\Omega)}\leqslant C_1\left(\gamma\int_{\Gamma_I}\theta_\nu(\gamma+2H)u'u-\int_{\Gamma_I}\theta_\nu\frac{\mathrm{d}u'}{\mathrm{d}s}\frac{\mathrm{d}u}{\mathrm{d}s}\right).$$

Moreover, by the a priori hypothesis (2.1) and the continuous embedding $H^2(\Gamma_I)\hookrightarrow C^1(\Gamma_I)$ (see for instance [1, Chap. 8]) we have that

$$\begin{aligned}
\|u'\|^2_{H^1(\Omega)} &\leqslant C\|u'\|_{C^1(\Gamma_I)}\left(\gamma\int_{\Gamma_I}|\theta_\nu(\gamma+2H)u'|+\int_{\Gamma_I}\left|\theta_\nu\frac{\mathrm{d}u'}{\mathrm{d}s}\right|\right) \\
&\leqslant C\left(\int_{\Gamma_I}|\theta_\nu(\gamma+2H)u'|+\int_{\Gamma_I}\left|\theta_\nu\frac{\mathrm{d}u'}{\mathrm{d}s}\right|\right).
\end{aligned}$$

Furthermore, by the Schwartz inequality and standard trace inequality we have that

$$\|u'\|^2_{H^1(\Omega)}\leqslant C\left(\frac{\|\theta\|^2_{L^2(\Gamma_I)}}{\varepsilon}+\varepsilon\|u'\|^2_{H^1(\Omega)}\right).$$

Hence, choosing $\varepsilon=\frac{1}{2C}$ (with the same C of the previous formula), we deduce that

$$\|u'\|^2_{H^1(\Omega)}\leqslant C\|\theta\|^2_{C_0^1(\Gamma_I)}.$$

Finally, by a standard trace inequality, we deduce that

$$\|u'\|^2_{H^{\frac{1}{2}}(\Gamma_A)}\leqslant C\|\theta\|^2_{C_0^1(\Gamma_I)}.$$

Hence $F':C_0^1(\Gamma_I)\rightarrow H^{\frac{1}{2}}(\Gamma_A)$ is bounded. The thesis follows immediately by the compact embedding

$$H^{\frac{1}{2}}(\Gamma_A)\hookrightarrow L^2(\Gamma_A),$$

see for instance [1, Chap. 8]. □

We observe that, in view of the above theorem, the issue of the identification of θ may be reformulated as the regularized inversion of a compact operator. Such kind of reformulation allows the use of singular values decomposition and the approximate inversion by the technique of Tikhonov regularization.

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