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Hardy-Poincaré inequalities with boundary singularities

Mouhamed Moustapha Fall* and Roberta Musina†

Abstract. Let Ω be a bounded domain in $\mathbb{R}^N$ with $0 \in \partial\Omega$ and $N \geq 2$. In this paper we study the Hardy-Poincaré inequality for maps in $H^1_0(\Omega)$. In particular we give sufficient and some necessary conditions so that the best constant is achieved.

Key Words: Hardy inequality, lack of compactness, nonexistence, supersolutions.

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Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, with $N \geq 2$. In this paper we assume that $0 \in \partial\Omega$ and we study the minimization problem

$$\mu_\lambda(\Omega) := \inf_{\substack{u \in H^1_0(\Omega) \\ u \neq 0}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx - \lambda \int_\Omega |u|^2 \, dx}{\displaystyle\int_\Omega |x|^{-2} |u|^2 \, dx}, \tag{0.1}$$

where $\lambda \in \mathbb{R}$ is a varying parameter. For $\lambda = 0$ the Ω-*Hardy constant* $\mu_0(\Omega) \geq (N-2)^2/4$ is the best constant in the Hardy inequality for maps supported by Ω. If $N = 2$ it has been proved in [4], Theorem 1.6, that $\mu_0(\Omega)$ is positive.

Problem (0.1) carries some similarities with the questions studied by Brezis and Marcus in [1], where the weight is the inverse-square of the distance from the boundary of Ω. Also the paper [5] by Dávila and Dupaigne is somehow related to the minimization problem (0.1). Indeed, notice that for any fixed $\lambda \in \mathbb{R}$, any extremal for $\mu_\lambda(\Omega)$ is a weak solution to the linear Dirichlet problem

$$\begin{cases} -\Delta u = \mu |x|^{-2} u + \lambda u & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.2}$$

where $\mu = \mu_\lambda(\Omega)$. If $\mu_\lambda(\Omega)$ is achieved, then $\mu_\lambda(\Omega)$ is the first eigenvalue of the operator $-\Delta - \lambda$ on $H^1_0(\Omega)$. Starting from a different point of view, for $0 \in \Omega$, $N \geq 3$ and $\mu \leq (N-2)^2/4$, Dávila and Dupaigne have proved in [5] the existence of the first eigenfunction φ_1 of the operator $-\Delta - \mu|x|^{-2}$ on a suitable functional space $H(\Omega) \supseteq H^1_0(\Omega)$. Notice that φ_1 solves (0.2), where the eigenvalue λ depends on the datum μ.

The problem of the existence of extremals for the Ω-Hardy constant $\mu_0(\Omega)$ was already discussed in [4] in case $N = 2$ (with Ω possibly unbounded or singular at $0 \in \partial\Omega$) and in [12], where Ω is a suitable compact perturbation of a cone in $\mathbb{R}^N$. Hardy-Sobolev inequalities with singularity at the boundary have been studied by several authors. We quote for instance [3], [6], [7], [8], [9], [10], and references there-in.

The minimization problem (0.1) is not compact, due to the group of dilations in $\mathbb{R}^N$. Actually it might happen that all minimizing sequences concentrate at 0. In this case $\mu_\lambda(\Omega)$ is not achieved and $\mu_\lambda(\Omega) = \mu^+$, where

$$\mu^+ = \frac{N^2}{4}$$

is the best constant in the Hardy inequality for maps with support in a half-space. Indeed in Section 2 we first show that

$$\sup_{\lambda \in \mathbb{R}} \mu_\lambda(\Omega) = \mu^+ \tag{0.3}$$

then we deduce that, provided $\mu_\lambda(\Omega) < \mu^+$, every minimizing sequence for $\mu_\lambda(\Omega)$ converges in $H^1_0(\Omega)$ to an extremal for $\mu_\lambda(\Omega)$.

We recall that Ω is said to be locally concave at $0 \in \partial\Omega$ if it contains a half-ball. That is there exists $r > 0$ such that

$$\{x \in \mathbb{R}^N \mid x \cdot \nu > 0\} \cap B_r(0) \subset \Omega\,, \tag{0.4}$$

where ν is the interior normal of $\partial\Omega$ at 0. Notice that if all the principal curvatures of $\partial\Omega$ at 0, with respect to ν, are strictly negative, then condition (0.4) is satisfied.

Our first main result is stated in the following theorem.

Theorem 0.1 *Let $\Omega \in \mathbb{R}^N$ be a smooth bounded domain with $0 \in \partial\Omega$. Assume that Ω is locally concave at 0. Then $\mu_\lambda(\Omega)$ is attained if and only if $\mu_\lambda(\Omega) < \mu^+$.*

The "only if" part, which is the most intriguing, is a consequence of Corollary 3.2 in Section 3, where we provide local nonexistence results for problem

$$\begin{cases} -\Delta u \geq \mu|x|^{-2}u + \lambda u & \text{on } \Omega \\ u \geq 0 & \text{in } \Omega, \end{cases} \tag{0.5}$$

also for negative values of the parameter λ.

Up to now several questions concerning the infimum $\mu_\lambda(\Omega)$ are still open. Put

$$\lambda^* := \inf \left\{ \lambda \in \mathbb{R} \mid \mu_\lambda(\Omega) < \mu^+ \right\}. \tag{0.6}$$

Since the map $\lambda \mapsto \mu_\lambda(\Omega)$ is non increasing, then $\mu_\lambda(\Omega)$ is achieved for any $\lambda > \lambda^*$, by the existence Theorem 2.2. If $\lambda^* \in \mathbb{R}$, then from (0.3) it follows that $\mu_\lambda(\Omega) = \mu^+$ for any $\lambda \leq \lambda^*$ and hence $\mu_\lambda(\Omega)$ is not achieved if $\lambda < \lambda^*$. We don't know if there exist domains Ω for which $\lambda^* = -\infty$. On the other hand we are able to prove the following facts (see Section 5 for the precise statements):

i) If Ω is locally convex at 0, that is, if there exists $r > 0$ such that $\Omega \cap B_r(0)$ is contained in a half-space, then $\lambda^* > -\infty$.

ii) If Ω is contained in a half-space then

$$\lambda^* \geq \frac{\lambda_1(\mathbb{D})}{|\mathrm{diam}(\Omega)|^2} \ , \tag{0.7}$$

where $\lambda(\mathbb{D})$ is the first Dirichlet eigenvalue of the unit ball $\mathbb{D}$ in $\mathbb{R}^2$ and $\mathrm{diam}(\Omega)$ is the diameter of Ω.

iii) For any $\delta > 0$ there exists $\rho_\delta > 0$ such that, if

$$\Omega \supseteq \{x \in \mathbb{R}^N \mid x \cdot \nu > -\delta|x| \ , \ \alpha < |x| < \beta \ \}$$

for some $\nu \in \mathbb{S}^{N-1}$, $\beta > \alpha > 0$ with $\beta/\alpha > \rho_\delta$, then $\lambda^* < 0$. In particular the Hardy constant $\mu_0(\Omega)$ is achieved.

The relevance of the geometry of Ω at the origin is confirmed by Theorem 0.1, by *i*) and by the existence theorems proved in [8], [9] and [10] for a related superlinear problem. However, it has to be noticed that also the (conformal) "size" of Ω (even far away from the origin) has some impact on the existence of compact minimizing sequences. Actually, no requirement on the curvature of Ω at 0 is needed in *iii*). In particular, there exist smooth domains having strictly positive principal curvatures at 0, and such that the Hardy constant $\mu_0(\Omega)$ is achieved.

The paper is organized as follows.

In Section 1 we point out few remarks on the Hardy inequality on dilation-invariant domains.

In Section 2, Theorem 2.2, we give sufficient conditions for the existence of minimizers for (0.1).

In Section 3 we prove some nonexistence theorems for solutions to (0.5) that might have an independent interest.

To prove inequality (0.7) in case Ω is contained in a half space, in Section 4 we provide computable remainder terms for the Hardy inequality on half-balls. We adopt here an argument by Brezis-Vázquez [2], where bounded domains $\Omega \subset \mathbb{R}^N$, with $N \geq 3$ and $0 \in \Omega$, are considered.

In Section 5 we estimate λ^* from below and form above, under suitable assumptions on Ω.

Notation

- $\mathbb{R}^N_+$ and $\mathbb{S}^{N-1}_+$ denote any half space and any hemisphere, respectively. More precisely,

$$\mathbb{R}^N_+ = \{x \in \mathbb{R}^N \mid x \cdot \nu > 0\ \}\ , \quad \mathbb{S}^{N-1}_+ = \mathbb{S}^{N-1} \cap \mathbb{R}^N_+$$

  where ν is any unit vector in $\mathbb{R}^N$.

- $B_R(x)$ is the open ball in $\mathbb{R}^N$ of radius r centered at x. If $x = 0$ we simply write B_R. If $N = 2$ we shall often write $\mathbb{D}_R$ and $\mathbb{D}$ instead of B_R, B_1, respectively.

- We denote by $H^1(\mathbb{S}^{N-1})$ the standard Sobolev space of maps on the unit sphere and by ∇_σ, Δ_σ the gradient and the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$, respectively.

- Let Σ be a domain in $\mathbb{S}^{N-1}$. We denote by $H^1_0(\Sigma)$ the closure of $C^\infty_c(\Sigma)$ in the $H^1(\mathbb{S}^{N-1})$-space and by $\lambda_1(\Sigma)$ the fist Dirichlet eigenvalue on Σ.

- For any domain $\Omega \subset \mathbb{R}^N$, we denote by $L^2(\Omega; |x|^{-2}\ dx)$ the space of measurable maps on Ω such that $\displaystyle\int_\Omega |x|^{-2}|u|^2\ dx < \infty$. We put also

$$\widehat{H}^1(\Omega) := H^1(\Omega) \cap L^2(\Omega; |x|^{-2}\ dx)\,,$$

  where $H^1(\Omega)$ is the standard Sobolev space of maps on Ω.

1 Preliminaries

In this section we collect a few remarks on the Hardy inequality on dilation-invariant domains that are partially contained for example in [4] (in case $N = 2$) and in [12].

Via polar coordinates, to any domain Σ in $\mathbb{S}^{N-1}$ we associate a cone $\mathcal{C}_\Sigma \subset \mathbb{R}^{N-1}$ and a (half) cylinder $\mathcal{Z}_\Sigma \subset \mathbb{R}^{N+1}$ by setting

$$\mathcal{C}_\Sigma := \{\ t\sigma \mid t > 0\ ,\ \sigma \in \Sigma\ \}\ , \quad \mathcal{Z}_\Sigma := \mathbb{R}_+ \times \Sigma\ .$$

If Σ is a smooth domain in $\mathbb{S}^{N-1}$, then $\mathcal{C}_\Sigma$ is a Lipschitz, dilation-invariant domain in $\mathbb{R}^{N-1}$. In particular, if Σ is a half-sphere, then $\mathcal{C}_\Sigma$ is a half-space. The map

$$\mathbb{R}^{N-1} \setminus \{0\} \to \mathbb{R}^{N+1}\ , \quad x \mapsto \left(-\log|x|, \frac{x}{|x|}\right)$$

is an homeomorphism $\mathcal{C}_\Sigma \to \mathcal{Z}_\Sigma$. It induces the Emden-Fowler transform

$$T : C_c^\infty(\mathcal{C}_\Sigma) \to C_c^\infty(\mathcal{Z}_\Sigma) \ , \quad u(x) = |x|^{\frac{2-N}{2}} \ (Tu) \left(-\log|x|, \frac{x}{|x|} \right) .$$

A direct computation based on the divergence theorem gives

$$\int_{\mathcal{C}_\Sigma} |\nabla u|^2 \ dx = \frac{(N-2)^2}{4} \int_0^\infty \int_\Sigma |Tu|^2 \ dsd\sigma + \int_0^\infty \int_\Sigma |\nabla_{s,\sigma} Tu|^2 \ dsd\sigma \tag{1.1}$$

$$\int_{\mathcal{C}_\Sigma} |x|^{-2}|u|^2 \ dx = \int_0^\infty \int_\Sigma |Tu|^2 \ dsd\sigma, \tag{1.2}$$

where $\nabla_{s,\sigma} = (\partial_s, \nabla_\sigma)$ denotes the gradient on $\mathbb{R}_+ \times \mathbb{S}^{N-1}$.

Now we introduce the Hardy constant on the cone $\mathcal{C}_\Sigma$:

$$\mu_0(\mathcal{C}_\Sigma) := \inf_{\substack{u \in C_c^\infty(\mathcal{C}_\Sigma) \\ u \neq 0}} \frac{\displaystyle\int_{\mathcal{C}_\Sigma} |\nabla u|^2 \ dx}{\displaystyle\int_{\mathcal{C}_\Sigma)} |x|^{-2}|u|^2 \ dx} . \tag{1.3}$$

In the next proposition we notice that the Hardy inequality on $\mathcal{C}_\Sigma$ is equivalent to the Poincaré inequality for maps supported be the cylinder $\mathcal{Z}_\Sigma$.

Proposition 1.1 *Let $\mathcal{C}_\Sigma$ be a cone. Then*

$$\mu_0(\mathcal{C}_\Sigma) = \frac{(N-2)^2}{4} + \lambda_1(\Sigma).$$

Proof. By (1.1), (1.2) it turns out that

$$\mu_0(\mathcal{C}_\Sigma) - \frac{(N-2)^2}{4} = \inf_{\substack{v \in C_c^\infty(\mathcal{Z}_\Sigma) \\ v \neq 0}} \frac{\displaystyle\int_0^\infty \int_\Sigma |\nabla_{s,\sigma} v|^2 \ dsd\sigma}{\displaystyle\int_0^\infty \int_\Sigma |v|^2 \ dsd\sigma} =: \lambda_1(\mathcal{Z}_\Sigma) .$$

The result follows by noticing that $\lambda_1(\mathcal{Z}_\Sigma) = \lambda_1(\Sigma)$. □

The eigenvalue $\lambda_1(\Sigma)$ is explicitly known in few cases. For example, if $\Sigma = \mathbb{S}^{N-1}_+$ is a half-sphere then $\lambda_1(\mathbb{S}^{N-1}_+) = N-1$. Thus, the Hardy constant of a half space is given by

$$\mu_0(\mathbb{R}^N_+) = \mu^+ := \frac{N^2}{4}\,. \tag{1.4}$$

If $N = 2$ and if $\mathcal{C}_{\Sigma_\theta} \subset \mathbb{R}^2$ is a cone of amplitude $\theta \in (0, 2\pi]$ then $\lambda_1(\Sigma_\theta)$ coincide with the Dirichlet eigenvalue on the interval $(0, \theta)$. Hence we get the conclusion, which was first pointed out in [4]:

$$\mu_0(\mathcal{C}_{\Sigma_\theta}) = \frac{\pi^2}{\theta^2} \geq \frac{1}{4}\,. \tag{1.5}$$

Let Σ be a domain in $\mathbb{S}^{N-1}$. If $N \geq 3$ the space $\mathcal{D}^{1,2}(\mathcal{C}_\Sigma)$ is defined in a standard way as a close subspace of $\mathcal{D}^{1,2}(\mathbb{R}^{N-1})$. Notice that in case $\Sigma = \mathbb{S}^{N-1}$ it turns out that

$$\mathcal{D}^{1,2}(\mathcal{C}_{\mathbb{S}^{N-1}}) = \mathcal{D}^{1,2}(\mathbb{R}^N \setminus \{0\}) = \mathcal{D}^{1,2}(\mathbb{R}^N)$$

by a known density result.

If $N = 2$ and if Σ is properly contained in $\mathbb{S}^1$, then $\mu_0(\mathcal{C}_\Sigma) > 0$ by (1.5). In this case we can introduce the space $\mathcal{D}^{1,2}(\mathcal{C}_\Sigma)$ by completing $C^\infty_c(\mathcal{C}_\Sigma)$ with respect to the Hilbertian norm $\left(\int_{\mathcal{C}_\Sigma} |\nabla u|^2 \, dx\right)^{1/2}$.

The next result is an immediate consequence of the fact that the Dirichlet eigenvalue problem of $-\Delta$ in the strip $\mathcal{Z}_\Sigma$ is never achieved. The same conclusion was already noticed in [4] in case $N = 2$ and in [12].

Proposition 1.2 *Let Σ be a domain in $\mathbb{S}^{N-1}$. Then $\mu_0(\mathcal{C}_\Sigma)$ is not achieved in $\mathcal{D}^{1,2}(\mathcal{C}_\Sigma)$.*

2 Existence

In this Section we show that the condition $\mu_\lambda(\Omega) < \mu^+ = N^2/4$ is sufficient to guarantee the existence of a minimizer for $\mu_\lambda(\Omega)$. We notice that throughout this section, the regularity of Ω can be relaxed to Lipschitz domains which are of class C^2 at 0. We start with a preliminary result.

Lemma 2.1 *Let Ω be a smooth domain with $0 \in \partial\Omega$. Then*

$$\sup_{\lambda \in \mathbb{R}} \mu_\lambda(\Omega) = \mu^+.$$

Proof. The proof will be carried out in two steps.
Step 1. We claim that $\sup_{\lambda\in\mathbb{R}} \mu_\lambda(\Omega) \geq \mu^+$.
We denote by ν the interior normal of $\partial\Omega$ at 0. For $\delta > 0$, we consider the cone

$$\mathcal{C}_-^\delta := \left\{ x \in \mathbb{R}^{N-1} \mid x \cdot \nu > -\delta |x| \right\}.$$

Now fix $\varepsilon > 0$. If δ is small enough then $\mu_0(\mathcal{C}_-^\delta) \geq \mu^+ - \varepsilon$. Since Ω is smooth at 0 then there exists a small radius $r > 0$ (depending on δ) such that $\Omega \cap B_{r_\delta}(0) \subset \mathcal{C}_-^\delta$.

Next, let $\psi \in C^\infty(B_r(0))$ be a cut-off function, satisfying

$$0 \leq \psi \leq 1 \ , \quad \psi \equiv 0 \text{ in } \mathbb{R}^N \setminus B_{\frac{r}{2}}(0) \ , \quad \psi \equiv 1 \text{ in } B_{\frac{r}{4}}(0) \, .$$

We write any $u \in H_0^1(\Omega)$ as $u = \psi u + (1 - \psi)u$, to get

$$\int_\Omega |x|^{-2} |u|^2 \, dx \leq \int_\Omega |x|^{-2} |\psi u|^2 \, dx + c \int_\Omega |u|^2 \, dx \, , \tag{2.1}$$

where the constant c do not depend on u. Since $\psi u \in \mathcal{D}^{1,2}(\mathcal{C}_-^\delta)$ then

$$(\mu^+ - \varepsilon) \int_\Omega |x|^{-2} |\psi u|^2 \, dx \leq \mu_0(\mathcal{C}_-^\delta) \int_\Omega |x|^{-2} |\psi u|^2 \, dx \leq \int_\Omega |\nabla(\psi u)|^2 \, dx \tag{2.2}$$

by our choice of the cone $\mathcal{C}_-^\delta$. In addition, we have

$$\int_\Omega |\nabla(\psi u)|^2 \, dx \leq \int_\Omega |\nabla u|^2 \, dx + \frac{1}{2} \int_\Omega \nabla(\psi^2) \cdot \nabla(u^2) \, dx + c \int_\Omega |u|^2 \, dx \, .$$

using integration by parts we get

$$\int_\Omega |\nabla(\psi u)|^2 \, dx \leq \int_\Omega |\nabla u|^2 \, dx - \frac{1}{2} \int_\Omega \Delta(\psi^2) |u|^2 \, dx + c \int_\Omega |u|^2 \, dx.$$

Comparing with (2.1) and (2.2) we infer that there exits a positive constant c depending only on δ such that

$$(\mu^+ - \varepsilon) \int_\Omega |x|^{-2} |u|^2 \, dx \leq \int_\Omega |\nabla u|^2 \, dx + c \int_\Omega |u|^2 \, dx \quad \forall u \in H_0^1(\Omega). \tag{2.3}$$

Hence we get $(\mu^+ - \varepsilon) \le \mu_{-c}(\Omega)$. Consequently $(\mu^+ - \varepsilon) \le \sup_\lambda \mu_\lambda(\Omega)$, and the conclusion follows by letting $\varepsilon \to 0$.

Step 2: We claim that $\sup_\lambda \mu_\lambda(\Omega) \le \mu^+$.

For $\delta > 0$ we consider the cone

$$\mathcal{C}_+^\delta := \left\{ x \in \mathbb{R}^{N-1} \mid x \cdot \nu > \delta |x| \right\}.$$

As in the first step, for any $\delta > 0$ there exists $r_\delta > 0$ such that $\mathcal{C}_+^\delta \cap B_r(0) \subset \Omega$ for all $r \in (0, r_\delta)$. Clearly by scale invariance, $\mu_0(\mathcal{C}_+^\delta \cap B_r(0)) = \mu_0(\mathcal{C}_+^\delta)$. For $\varepsilon > 0$, we let $\phi \in H_0^1(\mathcal{C}_+^\delta \cap B_r(0))$ such that

$$\frac{\displaystyle\int_{\mathcal{C}_+^\delta \cap B_r(0)} |\nabla \phi|^2 \, dx}{\displaystyle\int_{\mathcal{C}_+^\delta \cap B_r(0)} |x|^{-2} |\phi|^2 \, dx} \le \mu_0(\mathcal{C}_+^\delta) + \varepsilon.$$

From this we deduce that

$$\begin{aligned}
\mu_\lambda(\Omega) \quad &\le \quad \frac{\displaystyle\int_{\mathcal{C}_+^\delta \cap B_r(0)} |\nabla \phi|^2 \, dx - \lambda \int_{\mathcal{C}_+^\delta \cap B_r(0)} |\phi|^2 \, dx}{\displaystyle\int_{\mathcal{C}_+^\delta \cap B_{r_\delta}(0)} |x|^{-2} |\phi|^2 \, dx} \\
&\le \quad \mu_0(\mathcal{C}_+^\delta) + \varepsilon + |\lambda| \, \frac{\displaystyle\int_{\mathcal{C}_+^\delta \cap B_r(0)} |\phi|^2 \, dx}{\displaystyle\int_{\mathcal{C}_+^\delta \cap B_r(0)} |x|^{-2} |\phi|^2 \, dx}.
\end{aligned}$$

Since $\displaystyle\int_{\mathcal{C}_+^\delta \cap B_r(0)} |x|^{-2} |\phi|^2 \, dx \ge r^{-2} \int_{\mathcal{C}_+^\delta \cap B_r(0)} |\phi|^2 \, dx$, we get

$$\mu_\lambda(\Omega) \le \mu_0(\mathcal{C}_+^\delta) + \varepsilon + r^2 |\lambda|.$$

The conclusion follows immediately, since $\mu_0(\mathcal{C}_+^\delta) \to \mu^+$ when $\delta \to 0$. $\square$

Notice that if Ω is bounded then by (2.3) and Poincaré inequality

$$\mu_0(\Omega) > 0\,. \tag{2.4}$$

For $N = 2$ this was shown in [4] and for more general domains. We are in position to prove the main result of this section.

Theorem 2.2 *Let $\lambda \in \mathbb{R}$ and let Ω be a smooth bounded domain of $\mathbb{R}^N$ with $0 \in \partial\Omega$. If $\mu_\lambda(\Omega) < \mu^+$ then $\mu_\lambda(\Omega)$ is attained.*

Proof. Let $u_n \in H_0^1(\Omega)$ be a minimizing sequence for $\mu_\lambda(\Omega)$. We can normalize it to have

$$\int_\Omega |\nabla u_n|^2 = 1, \tag{2.5}$$

$$1 - \lambda \int_\Omega |u_n|^2 = \mu_\lambda(\Omega) \int_\Omega |x|^{-2}|u_n|^2 + o(1)\,. \tag{2.6}$$

We can assume that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, $|x|^{-1}u_n \rightharpoonup |x|^{-1}u$ weakly in $L^2(\Omega)$, and $u_n \to u$ in $L^2(\Omega)$, by (2.4) and by Rellich Theorem. Putting $\theta_n := u_n - u$, from (2.5) and (2.6) we get

$$\int_\Omega |\nabla\theta_n|^2 + \int_\Omega |\nabla u|^2 = 1 + o(1),$$

$$1 - \lambda \int_\Omega |u|^2 = \mu_\lambda(\Omega) \left(\int_\Omega |x|^{-2}|\theta_n|^2 + \int_\Omega |x|^{-2}|u|^2 \right) + o(1)\,. \tag{2.7}$$

By Lemma 2.1, for any fixed positive $\delta < \mu^+ - \mu_\lambda(\Omega)$, there exists $\lambda_\delta \in \mathbb{R}$ such that $\mu_{\lambda_\delta}(\Omega) \geq \mu^+ - \delta$. Hence

$$\int_\Omega |\nabla\theta_n|^2 + o(1) \geq (\mu^+ - \delta) \int_\Omega |x|^{-2}|\theta_n|^2\,,$$

as $\theta_n \to 0$ in $L^2(\Omega)$. Testing $\mu_\lambda(\Omega)$ with u we get

$$\begin{aligned}
\mu_\lambda(\Omega) \int_\Omega |x|^{-2}|u|^2 &\leq \int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2 \leq 1 - \int_\Omega |\nabla\theta_n|^2 - \lambda \int_\Omega |u|^2 + o(1) \\
&\leq 1 - (\mu^+ - \delta) \int_\Omega |x|^{-2}|\theta_n|^2 - \lambda \int_\Omega |u|^2 + o(1) \\
&\leq (\mu_\lambda(\Omega) - \mu^+ + \delta) \int_\Omega |x|^{-2}|\theta_n|^2 + \mu_\lambda(\Omega) \int_\Omega |x|^{-2}|u|^2 + o(1)
\end{aligned}$$

by (2.7). Therefore $\int_\Omega |x|^{-2}|\theta_n|^2 \to 0$, since $\mu_\lambda(\Omega) - \mu^+ + \delta < 0$. In particular,

$$\mu_\lambda(\Omega) \int_\Omega |x|^{-2}|u|^2 = \int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2$$

and $u \neq 0$ by (2.7). Thus u achieves $\mu_\lambda(\Omega)$. $\square$

We conclude this section with a corollary of Theorem 2.2.

Corollary 2.3 *Let Ω be a smooth bounded domain of $\mathbb{R}^N$ with $0 \in \partial\Omega$. Then*

$$\frac{(N-2)^2}{4} < \mu_0(\Omega) \le \frac{N^2}{4}\,.$$

Proof. It has been already proved in Lemma 2.1 that $\mu_\lambda(\Omega) \le \frac{N^2}{4}$. If the strict inequality holds, then there exists $u \in H^1_0(\Omega)$ that achieves $\mu_0(\Omega)$, by Theorem 2.2. But then $\frac{(N-2)^2}{4} < \mu_0(\Omega)$, otherwise a null extension of u outside Ω would achieve the Hardy constant on $\mathbb{R}^N$. $\square$

Remark 2.4 *Following [4], for non smooth domains Ω we can introduce the "limiting" Hardy constant*

$$\hat{\mu}_0(\Omega) = \sup_{r>0} \mu_0(\Omega \cap B_r)\,.$$

Using similar arguments it can be proved that $\sup_\lambda \mu_\lambda(\Omega) = \hat{\mu}_0(\Omega)$, and that $\mu_\lambda(\Omega)$ is achieved provided $\mu_\lambda(\Omega) < \hat{\mu}_0(\Omega)$.

3 Nonexistence

The main result in this section is stated in the following theorem.

Theorem 3.1 *Let Ω be a domain in $\mathbb{R}^N$, $N \ge 2$, and let $\lambda \in \mathbb{R}$. Assume that there exist $R > 0$ and a Lipschitz domain $\Sigma \subset \mathbb{S}^{N-1}$ such that $B_R \cap \mathcal{C}_\Sigma \subset \Omega$. If $u \in \widehat{H}^1(\Omega)$ solves*

$$\begin{cases} -\Delta u \ge \left(\dfrac{(N-2)^2}{4} + \lambda_1(\Sigma)\right)|x|^{-2}u + \lambda u & \text{in } \mathcal{D}'(\Omega \setminus \{0\}) \\ u \ge 0\,, \end{cases} \tag{3.1}$$

then $u \equiv 0$ in Ω.

Before proving Theorem 3.1 we point out some of its consequences.

Corollary 3.2 *Let Ω be a smooth bounded domain containing a half-ball and such that $0 \in \partial\Omega$. If $\mu_\lambda(\Omega) = \mu^+$ then $\mu_\lambda(\Omega)$ is not achieved.*

Proof. Assume that u achieves $\mu_\lambda(\Omega) = \mu^+$. Then u is a weak solution to

$$-\Delta u = \mu^+ |x|^{-2} u + \lambda u \,. \tag{3.2}$$

Test (3.2) with the negative and the positive part of u to conclude that u has constant sign. Now by the maximum principle $u > 0$ in Ω, contradicting Theorem 3.1, since $\Omega \supset B_R \cap \mathcal{C}_{\mathbb{S}^{N-1}_+}$ and $\lambda_1(\mathbb{S}^{N-1}_+) = N - 1$. □

We also point out the following consequence to Theorem 3.1, that holds for smooth domains Ω with $0 \in \partial\Omega$.

Theorem 3.3 *Let Ω be a smooth domain in $\mathbb{R}^N$, $N \geq 2$ with $0 \in \partial\Omega$ and let $\lambda \in \mathbb{R}$. If $u \in \widehat{H}^1(\Omega)$ solves*

$$\begin{cases} -\Delta u \geq \mu |x|^{-2} u + \lambda u & \text{in } \mathcal{D}'(\Omega) \\ u \geq 0 \end{cases}$$

for some $\mu > \mu^+$, then $u \equiv 0$ in Ω.

Proof. We start by noticing that there exists a geodesic ball $\Sigma \subset \mathbb{S}^{N-1}$ contained in a hemisphere, and such that $\lambda_1(\Sigma) \leq N - 1 + \mu - \mu^+$. Since $0 \in \partial\Omega$ and since $\partial\Omega$ is smooth then, up to a rotation, we can find a small radius $r > 0$ such that $B_r \cap \mathcal{C}_\Sigma \subset \Omega$. The conclusion follows from Theorem 3.1, as $\mu \geq (N-2)^2/4 + \lambda_1(\Sigma)$.
□

Remark 3.4 *Theorem 3.1 applies also when the origin lies in the interior of the domain. More precisely, let Ω be any domain in $\mathbb{R}^N$, with $N \geq 2$ and $0 \in \Omega$. If $u \in \widehat{H}^1_{\mathrm{loc}}(\Omega)$ is a nonnegative solution to*

$$-\Delta u \geq \frac{(N-2)^2}{4} \, |x|^{-2} u + \lambda u \qquad \text{in } \mathcal{D}'(\Omega \setminus \{0\})$$

for some $\lambda \in \mathbb{R}$, then $u \equiv 0$ in Ω.

In order to prove Theorem 3.1 we need few preliminary results about maps of two variables. Recall that $\mathbb{D}_R \subset \mathbb{R}^2$ is the open disk of radius R centered at 0.

Lemma 3.5 *Let $\psi \in \widehat{H}^1(\mathbb{D}_R)$ and $f \in L^1_{loc}(\mathbb{D}_R)$ for some $R > 0$. If ψ solves*

$$-\Delta\psi \geq f \quad \text{in } \mathcal{D}'(\mathbb{D}_R \setminus \{0\}) \tag{3.3}$$

then $-\Delta\psi \geq f$ in $\mathcal{D}'(\mathbb{D}_R)$.

Proof. We start by noticing that from

$$\infty > \int_{\mathbb{D}_R} |z|^{-2}|\psi|^2 = \int_0^R \frac{1}{r}\left(r^{-1}\int_{\partial B_r}|\psi|^2\right)$$

it follows that there exists a sequence $r_h \to 0$, $r_h \in (0, R)$ such that

$$r_h^{-1}\int_{\partial B_{r_h}}|\psi|^2 \to 0\ , \quad r_h^{-2}\int_{\partial B_{r_h^2}}|\psi|^2 \to 0 \tag{3.4}$$

as $h \to \infty$. Next we introduce the following cut-off functions:

$$\eta_h(z) = \begin{cases} 0 & \text{if } |z| \leq r_h^2 \\ \dfrac{\log|z|/r_h^2}{|\log r_h|} & \text{if } r_h^2 < |z| < r_h \\ 1 & \text{if } r_h \leq |z| \leq R. \end{cases}$$

Let $\varphi \in C_c^\infty(\mathbb{D}_R)$ be any nonnegative function. We test (3.3) with $\eta_h\varphi$ to get

$$\int \nabla\psi \cdot \nabla(\eta_h\varphi) \geq \int f\,\eta_h\varphi\ .$$

Since $\psi \in H^1(\mathbb{D}_R)$ and since $\eta_h \rightharpoonup 1$ weakly* in L^∞, it is easy to check that

$$\int f\,\eta_h\varphi = \int f\varphi + o(1)\ , \quad \int \eta_h\nabla\psi\cdot\nabla\varphi = \int \nabla\psi\cdot\nabla\varphi + o(1)$$

as $h \to \infty$. Therefore

$$\int \nabla\psi\cdot\nabla\varphi + \int \varphi\nabla\psi\cdot\nabla\eta_h \geq \int f\,\varphi + o(1)\,. \tag{3.5}$$

To pass to the limit in the left-hand side we notice that $\nabla\eta_h$ vanishes outside the annulus $A_h := \{r_h^2 < |z| < r_h\}$, and that η_h is harmonic on A_h. Thus

$$\begin{aligned} \int \varphi\nabla\psi\cdot\nabla\eta_h &= \int_{A_h}\nabla(\psi\varphi)\cdot\nabla\eta_h - \int_{A_h}\psi\nabla\varphi\cdot\nabla\psi \\ &= \mathcal{R}_h - \int_{A_h}\psi\nabla\varphi\cdot\nabla\eta_h \end{aligned}$$

where

$$\mathcal{R}_h := -r_h^{-2} \int_{\partial B_{r_h^2}} (\nabla\eta_h \cdot z)\psi\varphi + r_h^{-1} \int_{\partial B_{r_h}} (\nabla\eta_h \cdot z)\psi\varphi \,.$$

Now

$$|\mathcal{R}_h| \le c\,(r_h|\log r_h|)^{-1} \int_{\partial B_{r_h}} |\psi| + c\,(r_h^2|\log r_h|)^{-1} \int_{\partial B_{r_h^2}} |\psi|$$

where $c > 0$ is a constant that does not depend on h, and

$$(r_h|\log r_h|)^{-1} \int_{\partial B_{r_h}} |\psi| \le c\,|\log r_h|^{-1} \left(r_h^{-1} \int_{\partial B_{r_h}} |\psi|^2 \right)^{1/2} = o(1)$$

by Hölder inequality and by (3.4). In the same way, also

$$(r_h^2|\log r_h|)^{-1} \int_{\partial B_{r_h^2}} |\psi| \le c\,|\log r_h|^{-1} \left(r_h^{-2} \int_{\partial B_{r_h^2}} |\psi|^2 \right)^{1/2} = o(1)\,,$$

and hence $\mathcal{R}_h = o(1)$. Moreover from $\psi \in L^2(\mathbb{D}_R; |z|^{-2}\,dz)$ it follows that

$$\left| \int_{A_h} \psi\nabla\varphi \cdot \nabla\eta_h \right| \le |\log r_h|^{-1} \int |z|^{-1}\psi|\nabla\varphi| = o(1)\ .$$

In conclusion, we have proved that $\displaystyle\int \varphi\nabla\psi \cdot \nabla\eta_h = o(1)$ and therefore (3.5) gives

$$\int \nabla\psi \cdot \nabla\varphi \ge \int f\,\varphi\,.$$

Since φ was an arbitrary nonnegative function in $C_c^\infty(\mathbb{D}_R)$, this proves that $-\Delta\psi \ge f$ in the distributional sense on $\mathbb{D}_R$, as desired. □

The same proof gives a similar result for subsolutions.

Lemma 3.6 *Let $\varphi \in \widehat{H}^1(\mathbb{D}_R)$ and $f \in L^1_{loc}(\mathbb{D}_R)$ for some $R > 0$. If φ solves*

$$\Delta\varphi \ge f \qquad \text{in } \mathcal{D}'(\mathbb{D}_R \setminus \{0\})$$

then $\Delta\varphi \ge f$ in $\mathcal{D}'(\mathbb{D}_R)$.

The next result is crucial in our proof. We state it is a more general form than needed, as it could have an independent interest. Notice that we do not need any a priori knowledge of the sign of ψ in the interior of its domain.

Lemma 3.7 *For any $\lambda \in \mathbb{R}$ there exists $R_\lambda > 0$ such that for any $R \in (0, R_\lambda)$, $\varepsilon > 0$, problem*

$$\begin{cases} -\Delta\psi \geq \lambda\psi & \text{in } \mathcal{D}'(\mathbb{D}_R \setminus \{0\}) \\ \psi \geq \varepsilon & \text{on } \partial\mathbb{D}_R. \end{cases} \tag{3.6}$$

has no solution $\psi \in \widehat{H}^1(\mathbb{D}_R)$.

Proof. We fix $R_\lambda < 1/3$ small enough, in such a way that

$$\lambda < \lambda_1(\mathbb{D}_{R_\lambda}) \quad \text{if } \lambda \geq 0\,, \tag{3.7}$$

$$|\lambda||z|^2\,|\log|z||^2 \leq \frac{3}{4} \quad \text{for any } z \in \mathbb{D}_{R_\lambda}\,,\ \text{if } \lambda < 0\,, \tag{3.8}$$

We claim that the conclusion in Lemma 3.7 holds with this choice of R_λ. We argue by contradiction. Let $R < R_\lambda$ and $\varepsilon > 0$, $\psi \in \widehat{H}^1(\mathbb{D}_R)$ as in (3.6).

For any $\delta \in (1/2, 1)$ we introduce the following radially symmetric function on $\mathbb{D}_R$:

$$\varphi_\delta(z) = |\log|z||^{-\delta}\ .$$

By direct computation one can easily check that $\varphi_\delta \in \widehat{H}^1(\mathbb{D}_R)$, and in particular

$$(2\delta - 1)\int_{\mathbb{D}_R} |z|^{-2}|\varphi_\delta|^2 = 2\pi + o(1) \quad \text{as } \delta \to \frac{1}{2}. \tag{3.9}$$

Since $\delta > 1/2$ then φ_δ is a smooth solution to

$$\Delta\varphi_\delta \geq \frac{3}{4}\,|z|^{-2}\,|\log|z||^{-2+\delta} = \frac{3}{4}\,|z|^{-2}\,|\log|z||^{-2}\,\varphi_\delta \tag{3.10}$$

in $\mathbb{D}_R \setminus \{0\}$. By Lemma 3.6 we infer that φ_δ solves (3.10) in the dual of $\widehat{H}^1(\mathbb{D}_R)$. Next we put

$$v := \varepsilon\varphi_\delta - \psi \in \widehat{H}^1(\mathbb{D}_R)\,,$$

and we notice that $v \leq 0$ on $\partial\mathbb{D}_R$, as $R < 1/3$. Notice also that

$$\begin{aligned} \Delta v &\geq \frac{3}{4}\,|z|^{-2}\,|\log|z||^{-2}\,(\varepsilon\varphi_\delta) + \lambda\psi \\ &= \left[\frac{3}{4}\,|z|^{-2}\,|\log|z||^{-2} + \lambda\right](\varepsilon\varphi_\delta) - \lambda v \end{aligned}$$

on the dual of $\widehat{H}^1(\mathbb{D}_R)$, by (3.8). We use as test function $v^+ := \max\{v, 0\} \in H^1_0(\mathbb{D}_R) \cap \widehat{H}^1(\mathbb{D}_R)$ to get

$$-\int_{\mathbb{D}_R} |\nabla v^+|^2 \geq \int_{\mathbb{D}_R} \left[\frac{3}{4}\,|z|^{-2}\,|\log|z||^{-2} + \lambda\right](\varepsilon\varphi_\delta)v^+ - \lambda \int_{\mathbb{D}_R} |v^+|^2 .$$

If $\lambda \geq 0$ we infer that

$$\int_{\mathbb{D}_R} |\nabla v^+|^2 \leq \lambda \int_{\mathbb{D}_R} |v^+|^2$$

and hence $v^+ \equiv 0$ on $\mathbb{D}_R$ by (3.7). If $\lambda < 0$ we get

$$0 \geq -\int_{\mathbb{D}_R} |\nabla v^+|^2 \geq |\lambda| \int_{\mathbb{D}_R} |v^+|^2 ,$$

hence again $v^+ = 0$ on $\mathbb{D}_R$, by (3.8). Thus $\psi \geq \varepsilon\varphi_\delta$ on $\mathbb{D}_R$ and therefore

$$\infty > \int_{\mathbb{D}_R} |z|^{-2}|\psi|^2 \geq \varepsilon \int_{\mathbb{D}_R} |z|^{-2}|\varphi_\delta|^2 ,$$

which contradicts (3.9). □

Proof of Theorem 3.1. Without loss of generality, we may assume that $\lambda < 0$. Let $\Phi > 0$ be the first eigenfunction of $-\Delta_\sigma$ on Σ. Thus Φ solves

$$\begin{cases} -\Delta_\sigma \Phi = \lambda_1(\Sigma)\Phi & \text{in } \Sigma \\ \Phi = 0\ , \quad \dfrac{\partial\Phi}{\partial\eta} \leq 0 & \text{on } \partial\Sigma, \end{cases} \tag{3.11}$$

where $\eta \in T_\sigma(\mathbb{S}^{N-1})$ is the exterior normal to Σ at $\sigma \in \partial\Sigma$.

By density and the trace theorem, we can define the radially symmetric map ψ in $\mathbb{D}_R \setminus \{0\}$ as

$$\psi(z) = |z|^{\frac{N-2}{2}} \int_\Sigma u(|z|\sigma)\Phi(\sigma)\,d\sigma \;= |z|^{\frac{N-2}{2}} \int_{|z|\Sigma} u(\sigma')\Phi_{|z|}(\sigma')\,d\sigma', \tag{3.12}$$

where $\Phi_r(\sigma') = \Phi(\frac{\sigma'}{r})$ for all $\sigma' \in r\Sigma$. Since in polar coordinates $(r, \sigma) \in (0, \infty) \times \mathbb{S}^{N-1}$ it holds that

$$u_{rr} = -(N-1)r^{-1}u_r - r^{-2}\Delta_\sigma u\ ,$$

direct computations based on (3.1) lead to

$$-\Delta\psi \geq \lambda\psi \ \text{ in } \mathcal{D}'(\mathbb{D}_R \setminus \{0\}).$$

We claim that $\psi \in \widehat{H}^1(\mathbb{D}_R)$. Indeed, for $r = |z|$,

$$|\psi'| \le cr^{\frac{N-2}{2}-1} \int_\Sigma |u(r\sigma)| + cr^{\frac{N-2}{2}} \int_\Sigma |\nabla u(r\sigma)| \ ,$$

and, by Hölder inequality,

$$\int_{\mathbb{D}_R} \left(r^{\frac{N-2}{2}-1} \int_\Sigma |u(r\sigma)| \right)^2 = c \int_0^R \int_\Sigma r^{N-3} u^2 \le c \int_\Omega |x|^{-2} u^2 < \infty \ ,$$

$$\int_{\mathbb{D}_R} \left(r^{\frac{N-2}{2}} \int_\Sigma |\nabla u(r\sigma)| \right)^2 \le c \int_0^R r^{N-1} \int_\Sigma |\nabla u|^2 \le c \int_\Omega |\nabla u|^2 < \infty.$$

Finally, $\psi \in L^2(R^2_R; |z|^{-2} dz)$ as

$$\int_{\mathbb{D}_R} |z|^{-2} |\psi|^2 = 2\pi \int_0^R r^{-1} |\psi|^2 \le c \int_0^R r^{N-3} \int_\Sigma |u|^2 = c \int_\Omega |x|^{-2} |u|^2 < \infty \, .$$

Thus Lemma 3.7 applies and since ψ is radially symmetric we get $\psi \equiv 0$ in a neighborhood of 0. Hence $u \equiv 0$ in $B_r \cap \mathcal{C}_\Sigma$, for $r > 0$ small enough. To conclude the proof in case Ω strictly contains $B_r \cap \mathcal{C}_\Sigma$, take any domain Ω' compactly contained in $\Omega \setminus \{0\}$ and such that Ω' intersects $B_r \cap \mathcal{C}_\Sigma$. Via a convolution procedure, approximate u in $H^1(\Omega')$ by a sequence of smooth maps u_ε that solve

$$-\Delta u_\varepsilon + |\lambda| u_\varepsilon \ge 0 \quad \text{in } \Omega' \, .$$

Since $u_\varepsilon \ge 0$ and $u_\varepsilon \equiv 0$ on $\Omega' \cap B_r \cap \mathcal{C}_\Sigma$, then $u_\varepsilon \equiv 0$ on Ω' by the maximum principle. Thus also $u \equiv 0$ in Ω', and the conclusion follows. $\square$

4 Remainder terms

We prove here some inequalities that will be used in the next section to estimate the infimum λ^* defined in (0.6).

Brezis and Vázquez proved in [2] the following improved Hardy inequality:

$$\int_\Omega |\nabla u|^2 - \frac{(N-2)^2}{4} \int_\Omega |x|^{-2} |u|^2 \ge \omega_N \frac{\lambda(\mathbb{D})}{|\Omega|} \int_\Omega |u|^2 \, , \tag{4.1}$$

that holds for any $u \in C^\infty_c(\Omega)$. Here $\Omega \subset \mathbb{R}^N$ is any bounded domain, $\lambda(\mathbb{D})$ is the first Dirichlet eigenvalue of the unit ball $\mathbb{D}$ in $\mathbb{R}^2$, and ω_N, $|\Omega|$ denote the measure of the unit ball in $\mathbb{R}^N$ and of Ω, respectively. If $0 \in \Omega$ then $(N-2)^2/4$ is the Hardy constant $\mu_0(\Omega)$ relative to the domain Ω, by the invariance of the ratio

$$\frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx}{\displaystyle\int_\Omega |x|^{-2}|u|^2 \, dx}$$

with respect to dilations in $\mathbb{R}^N$.

We show that a Brezis-Vázquez type inequality holds in case the singularity is placed at the boundary of the domain. We start with conic domains

$$\mathcal{C}_{R,\Sigma} = \{t\sigma \mid t \in (0,R),\, \sigma \in \Sigma\,\},$$

where $\Sigma \subset \mathbb{S}^{N-1}$ and $R > 0$.

Proposition 4.1 *Let Σ be a domain in $\mathbb{S}^{N-1}$. Then*

$$\int_{\mathcal{C}_{R,\Sigma}} |\nabla u|^2 - \mu_0\left(\mathcal{C}_\Sigma\right) \int_{\mathcal{C}_{R,\Sigma}} |x|^{-2}|u|^2 \geq \frac{\lambda_1(\mathbb{D})}{R^2} \int_{\mathcal{C}_{R,\Sigma}} |u|^2, \qquad \forall u \in C^\infty_c(\mathcal{C}_{1,\Sigma}). \tag{4.2}$$

Proof. By homogeneity, it suffices to prove the proposition for $R = 1$. Fix $u \in C^\infty_c(\mathcal{C}_{1,\Sigma})$ and compute in polar coordinates $t = |x|$, $\sigma = x/|x|$:

$$\int_{\mathcal{C}_{1,\Sigma}} |\nabla u|^2 = \int_0^1 \int_\Sigma \left|\frac{\partial u}{\partial t}\right|^2 t^{N-1} dt d\sigma + \int_0^1 \int_\Sigma |\nabla_\sigma u|^2 \, t^{N-3} dt d\sigma\,,$$

$$\int_{\mathcal{C}_{1,\Sigma}} |x|^{-2}|u|^2 = \int_0^1 \int_\Sigma |u|^2 t^{N-3} dt d\sigma.$$

Since for every $t \in (0,1)$ it holds that

$$\int_\Sigma |\nabla_\sigma u|^2 \, t^{N-3} d\sigma \geq \lambda_1(\Sigma) \int_\Sigma |u|^2 t^{N-3} d\sigma,$$

then by Proposition 1.1 we only have to show that

$$\int_0^1 \left|\frac{\partial u}{\partial t}\right|^2 t^{N-1} dt - \frac{(N-2)^2}{4} \int_0^1 |u|^2 t^{N-3} dt \geq \lambda_1(\mathbb{D}) \int_0^1 |u|^2 t^{N-1} dt \tag{4.3}$$

for any fixed $\sigma \in \Sigma$. For that, we put $w(t) = t^{\frac{N-2}{2}} u(t\sigma)$, and we compute

$$\begin{aligned}
\int_0^1 \left|\frac{\partial u}{\partial t}\right|^2 t^{N-1} dt \quad &- \quad \mu_0\left(\mathbb{R}^N\right) \int_0^1 |u|^2 t^{N-3} dt \\
&= \int_0^1 \left|\frac{\partial w}{\partial t}\right|^2 t dt + (2-N) \int_0^1 \frac{\partial w}{\partial t} w dt \\
&= \int_0^1 \left|\frac{\partial w}{\partial t}\right|^2 t dt + \frac{(2-N)}{2} \int_0^1 \frac{\partial w^2}{\partial t} dt = \int_0^1 \left|\frac{\partial w}{\partial t}\right|^2 t dt \\
&\geq \lambda_1(\mathbb{D}) \int_0^1 w^2 t dt = \lambda_1(\mathbb{D}) \int_0^1 |u|^2 t^{N-1} dt.
\end{aligned}$$

This gives (4.3) and the proposition is proved. □

The main result in this section is contained in the next theorem.

Theorem 4.2 *Let Ω be a bounded domain of $\mathbb{R}^N$ with $0 \in \partial\Omega$. If Ω is contained in a half-space then*

$$\int_\Omega |\nabla u|^2 - \mu^+ \int_\Omega |x|^{-2}|u|^2 \geq \frac{\lambda_1(\mathbb{D})}{|diam(\Omega)|^2} \int_\Omega |u|^2 \qquad \forall u \in H_0^1(\Omega).$$

Proof. Let $R > 0$ be the diameter of Ω. Then $\Omega \subset B_R^+$, where B_R^+ is a half ball of radius R centered at the origin. Take Σ to be a half sphere in $\mathbb{S}^{N-1}$ in Proposition 4.1, so that $\mathcal{C}_\Sigma$ is an half-space. Recalling (1.4), we conclude that

$$\int_{B_R^+} |\nabla u|^2 - \mu^+ \int_{B_R^+} |x|^{-2}|u|^2 \geq \frac{\lambda_1(\mathbb{D})}{R^2} \int_{B_R^+} |u|^2$$

for any $R > 0$, $u \in C_c^\infty(\Omega)$ and the theorem readily follows. □

Remark 4.3 *Let Ω be a bounded domain of $\mathbb{R}^2$ with $0 \in \partial\Omega$ and assume that Ω does not intersect an half-line emanating from the origin. Then (1.5) and Proposition 4.1 imply the following improved Hardy inequality:*

$$\int_\Omega |\nabla u|^2 - \frac{1}{4} \int_\Omega |x|^{-2}|u|^2 \geq \frac{\lambda_1(\mathbb{D})}{|diam(\Omega)|^2} \int_\Omega |u|^2 \qquad \forall u \in H_0^1(\Omega).$$

Remark 4.4 *As pointed out by Brezis-Vázquez in [2], Extension 4.3, the following Hardy-Sobolev inequality holds*

$$\int_{\mathcal{C}_{1,\Sigma}} |\nabla u|^2 - \mu_0\left(\mathcal{C}_{1,\Sigma}\right) \int_{\mathcal{C}_{1,\Sigma}} |x|^{-2}|u|^2 \geq c_p \left(\int_{\mathcal{C}_{1,\Sigma}} |u|^p\right)^{\frac{2}{p}}, \qquad \forall u \in C_c^\infty(\mathcal{C}_{1,\Sigma})$$

for all $p \in \left(2, \frac{2N}{N-2}\right)$, *where* c_p *is a positive constant depending on* p *and* N.

5 Estimates on λ^*

In this section we provide sufficient conditions to have $\lambda^* > -\infty$ or $\lambda^* < 0$.

5.1 Estimates from below

Let Ω be a smooth domain in $\mathbb{R}^N$ with $0 \in \partial\Omega$. We say that Ω is *locally convex* at 0 if exists a ball B centered at 0 such that $\Omega \cap B$ is contained in a half-space. In essence, for domains of class C^2 this means that all the principal curvatures of $\partial\Omega$ (with respect to the interior normal) at 0 are strictly positive.

In the case where Ω is locally convex at $0 \in \partial\Omega$, the supremum in Lemma 2.1 is attained.

Proposition 5.1 *If* Ω *is locally convex at* 0, *then there exists* $\lambda^*(\Omega) \in \mathbb{R}$ *such that*

$$\mu_\lambda(\Omega) = \mu^+, \qquad \forall \lambda \leq \lambda^*(\Omega),$$
$$\mu_\lambda(\Omega) < \mu^+, \qquad \forall \lambda > \lambda^*(\Omega).$$

Proof. The locally convexity assumption at 0 means that there exists $r > 0$ such that $B_r(0) \cap \Omega$ is contained in a half space. We let $\psi \in C_c^\infty(\mathbb{R}^N)$ with $0 \leq \psi \leq 1$, $\psi \equiv 0$ in $\mathbb{R}^N \setminus B_{\frac{r}{2}}(0)$ and $\psi \equiv 1$ in $B_{\frac{r}{4}}(0)$. Arguing as in the proof of Lemma 2.1, for every $u \in H_0^1(\Omega)$ we get

$$\int_\Omega |x|^{-2}|u|^2\, dx \leq \int_\Omega |x|^{-2}|\psi u|^2\, dx + c\int_\Omega |u|^2\, dx \tag{5.1}$$

for some constant $c = c(r) > 0$. Since $\psi u \in H_0^1(B_r(0) \cap \Omega)$, from the definition of μ^+ we infer

$$\mu^+ \int_\Omega |x|^{-2}|\psi u|^2\, dx \leq \int_\Omega |\nabla(\psi u)|^2\, dx.$$

As in Lemma 2.1 we get

$$\int_\Omega |\nabla(\psi u)|^2 \, dx \le \int_\Omega |\nabla u|^2 \, dx + c \int_\Omega |u|^2 \, dx \, .$$

Comparing with (5.1), we infer that there exits a positive constant c such that

$$\mu^+ \int_\Omega |x|^{-2} |u|^2 \, dx \le \int_\Omega |\nabla u|^2 \, dx + c \int_\Omega |u|^2 \, dx.$$

This proves that $\mu_{-c}(\Omega) \ge \mu^+$. Thus $\mu_{-c}(\Omega) = \mu^+$ by Lemma 2.1. Finally, noticing that $\mu_\lambda(\Omega)$ is decreasing in λ, we can set

$$\lambda^*(\Omega) := \sup\{\lambda \in \mathbb{R} \quad : \quad \mu_\lambda(\Omega) = \mu^+\} \tag{5.2}$$

so that $\mu_\lambda(\Omega) < \mu^+$ for all $\lambda > \lambda^*(\Omega)$. □

Finally, we notice that by Lemma 2.1, if Ω is contained in a half-space, then $\mu_0(\Omega) = \mu^+$, and therefore $\lambda^*(\Omega) \ge 0$. Thus, From Theorem 4.2 we infer the following result.

Theorem 5.2 *Let Ω be a bounded smooth domain with $0 \in \partial\Omega$. If Ω is contained in a half-space then*

$$\lambda^*(\Omega) \ge \frac{\lambda_1(\mathbb{D})}{|diam(\Omega)|^2} \, .$$

It would be of interest to know if it is possible to get lower bounds depending only on the measure of Ω, as in [2] and [11].

5.2 Estimates from above

The local convexity assumption of Ω at 0 does not necessary implies that $\lambda^*(\Omega) \ge 0$. Indeed the following remark holds.

Proposition 5.3 *For any $\delta > 0$ there exists $\rho_\delta > 0$ such that if Ω is a smooth domain with $0 \in \partial\Omega$ and*

$$\Omega \supseteq \{x \in \mathbb{R}^N \mid x \cdot \nu > -\delta |x| \ , \ \alpha < |x| < \beta \ \}$$

for some $\nu \in \mathbb{S}^{N-1}$, $\beta > \alpha > 0$ with $\beta/\alpha > \rho_\delta$, then $\lambda^* < 0$. In particular the Hardy constant $\mu_0(\Omega)$ is achieved.

Proof. Since the cone

$$\mathcal{C}_\delta = \{x \in \mathbb{R}^N \mid x \cdot \nu > -\delta|x| \ \}$$

contains a hemispace, then its Hardy constant is smaller than μ^+. Thus there exists $u \in C_c^\infty(\mathcal{C}_\delta)$ such that

$$\frac{\displaystyle\int_{\mathcal{C}_\delta} |\nabla u|^2 \ dx}{\displaystyle\int_{\mathcal{C}_\delta} |x|^{-2}|u|^2 \ dx} < \mu^+ \ .$$

Assume that the support of u is contained in an annulus of radii $b > a > 0$. Then the conclusion in Proposition 5.3 holds, with $\rho := b/a$. □

Notice that Ω can be locally strictly convex at 0.

Remark 5.4 *A similar remark holds for the following minimization problem, which is related to the Caffarelli-Kohn-Nirenberg inequalities:*

$$\inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \ dx}{\left(\displaystyle\int_\Omega |x|^{-b}|u|^p \ dx\right)^{2/p}} , \tag{5.3}$$

where $2 < p < 2^*$, $b := N - p(N-2)/2$. *In case* $0 \in \partial\Omega$, *the minimization problem (5.3) was studied in [8], [9] and [10].*

Remark 5.5 *We do not know wether the strict local concavity of* Ω *at 0 can implies that* $\mu_0(\Omega) < \mu^+$. *See the paper [8] by Ghoussoub and Kang for the minimization problem (5.3).*

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