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Abstract

Let (m_n) be a faithfully enumerated sequence of integers with $m_n|m_{n+1}$ for every $n \in \mathbb{N}$. We describe the topologically (m_n)-torsion elements of the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (written additively), namely those elements $x \in \mathbb{T}$ such that $m_n x$ coverges to 0.

1 Introduction

Definition 1.1. [8, 9, 20] An element x of an abelian topological group G is:

(a) *topologically torsion* if $n!x \to 0$;

(b) *topologically p-torsion*, for a prime p, if $p^n x \to 0$.

Clearly, every (p-)torsion element is topologically (p-)torsion. Armacost [1] showed that the topologically p-torsion elements of the the circle group $\mathbb{T} = (\mathbb{R}/\mathbb{Z}, +)$ are precisely the p-torsion ones. On the other hand, he found a non-torsion, topologically torsion element of $\mathbb{T}$. The topologically torsion elements of $\mathbb{T}$ form a subgroup of $\mathbb{T}$, which will be denoted it by $\mathbb{T}\,!$, following [1]. The problem to find an explicit description of $\mathbb{T}\,!$ was set by Armacost [1, p. 34]. A partial solution was given in [15, Chap.4] and a final solution was given by Borel [7] who was unaware of that partial solution. Later, a solution was obtained also in [12] as a particular case of a more general problem (these authors were, in turn, unaware of the paper [7]).

The more general problem, faced in [15, §4.4.2] and [12], is based on the following more general notion, proposed in [15, §4.4.2] (see also [11]) in order to unify the notions of topologically p-torsion element and topologically torsion element:

Definition 1.2. For an abelian topological group G and a sequence of natural numbers $\underline{m} = (m_n)$ with

$$1 < m_1 < m_2 < \ldots < m_n < \ldots \quad \text{and } m_n|m_{n+1} \text{ for every } n \in \mathbb{N}. \tag{1}$$

callan element $x \in G$ *topologically $\underline{m}$-torsion* if $m_n x \to 0$ in G.

For a prime p and the sequence $\underline{p} = (p^n)$ one obtains (a) from Definition 1.1, while (b) from Definition 1.1 is obtained with $m_n = n!$. For an abelian topological group G the subset of all topologically $\underline{m}$-torsion elements of G,

$$t_{\underline{m}}(G) := \{x \in G : m_n x \to 0\}, \tag{2}$$

is a subgroup of G. Clearly, Definition 1.2 and (2) can be considered also for sequences $\underline{m} = (m_n)$ where $m_n|m_{n+1}$ fails. The subgroups of $\mathbb{T}$ of the form $t_{\underline{m}}(\mathbb{T})$, named *characterized*, were extensively studied in this more general setting ([2, 3, 4, 11]). In [5] the countable non-torsion subgroups of $\mathbb{T}$ were shown to be characterized. Appropriate extension of characterized subgroups for arbitrary topological abelian groups was proposed in [14]. In [13] it was shown that every countable subgroup of a compact metrizable abelian group is characterized. Some new necessary conditions were found for the characterized subgroups of the compact metrizable abelian groups were obtained in [16, 17].

A further progress towards Armacost's problem was obtained in [12, Theorem 2.2] (see also the survey [11]), where a solution of the problem of description of the subgroup $t_{\underline{m}}(\mathbb{T})$ was claimed for sequences of the form (1).

Recently the second named author of the present paper discovered a gap in [12, Theorem 2.2]. The main goal of this paper is to give a complete solution of this problem. We also show that Corollary 2.4 and Corollary 2.6 of [12] are not affected by this gap, by providing correct proofs. Finally, we provide in §4 counter-examples showing that Theorem 2.2 and corollaries 2.3 and 2.5 from [12] are false.

Notation. The symbols $\mathbb{P}$, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are used for the set of primes, the set of naturals, the group of integers and the group of rationals, respectively. The circle group $\mathbb{T}$ is identified with the quotient group $\mathbb{R}/\mathbb{Z}$ of the reals $\mathbb{R}$ and carries its usual compact topology. For $x \in \mathbb{R}$ we denote by $\{x\}$ the difference $x - [x]$.

Let G be an abelian group. The cyclic subgroup of G generated by $g \in G$ is denoted by $\langle g \rangle$. The cyclic group of order n is denoted by $\mathbb{Z}(n)$. For $n \in \mathbb{N}$ we put $nG = \{ng \in G : g \in G\}$. We say that G is *divisible* if $G = pG$ for every prime p.

The symbol $\mathfrak{c}$ stands for the cardinality of the continuum, so $\mathfrak{c} = 2^{\aleph_0}$. For undefined terms see [15, 18].

2 Main results

2.1 Background on the solution of Armacost's problem

The solution of Armacost's problem requires the following representation of reals $x \in [0, 1)$. There exist integers $0 \leq c_n < n$ with

$$x = \sum_{n=2}^{\infty} \frac{c_n}{n!}, \tag{3}$$

such that $c_n < n - 1$ for infinitely many n. The integers c_n are uniquely determined by these properties. In these terms the description of the topologically torsion elements of $\mathbb{T}$ is the following:

Theorem 2.1. *For $x \in [0, 1)$ with representation (3), $\varphi(x) \in \mathbb{T}!$ if and only if*

$$\lim_n \varphi\left(\frac{c_n}{n}\right) = 0 \text{ in } \mathbb{T}. \tag{4}$$

This theorem follows directly from item (b) in Theorem 2.3.

In order to describe the subgroup $t_{\underline{m}}(\mathbb{T}) := \{x \in \mathbb{T} \mid \lim_{n\in\mathbb{N}} m_n x = 0\}$ for a sequence $\underline{m}$ of natural numbers as in (1), we need a similar representation as above. Let $b_1 = m_1$ and note that $b_n = \frac{m_n}{m_{n-1}} \in \mathbb{Z}$ and $b_n > 1$ for every $n > 1$. Clearly, $m_n = b_1 \ldots b_n$ for every $n \in \mathbb{N}$. Then again for every $x \in [0, 1)$ one can build a unique sequence of integers (c_n) such that $0 \leq c_n < b_n$ for every n,

$$x = \sum_{n=1}^{\infty} \frac{c_n}{m_n}, \tag{5}$$

and $c_n < b_n - 1$ for infinitely many n. This can be done as follows. Let $c_1 = [m_1 x]$, so that $x_1 = x - c_1/m_1 < 1/m_1 = b_2/m_2$. Suppose that $c_1, \ldots c_k$ are defined for some $k \geq 1$ such that for $x_k = \sum_{n=1}^{k} \frac{c_n}{m_n}$ one has $x - x_k < 1/m_k$. Let $c_{k+1} := [m_{k+1}(x - x_k)]$. Then $c_{k+1} < b_{k+1}$.) We shall refer to the representation (5) as *canonical representation* of x. Clearly, in these terms "topologically torsion" is obtained as "topologically $\underline{m}$-torsion" for the shifted sequence $m_n = (n + 1)!$.

Let (5) $\text{supp}(x) = \{n \in \mathbb{N} \mid c_n \neq 0\}$ and $\text{supp}_b(x) = \{n \in \mathbb{N} \mid c_n = b_n - 1\}$ where $x \in [0, 1)$ with canonical representation.

Notation. Call an infinite set A of naturals

- *b-bounded* if the sequence $\{b_n : n \in A\}$ is bounded.
- *b-divergent* if the sequence $\{b_n : n \in A\}$ diverges to infinity.

Remark 2.2. We already defined $t_{\underline{m}}(\mathbb{T})$, now if $A \in [\mathbb{N}]^{\aleph_0}$ let $t_{\underline{m}_A}(\mathbb{T}) = \{z \in \mathbb{T} \mid \lim_{n\in A} m_n z = 0\}$. For all $A \in [\mathbb{N}]^{\aleph_0}$ we have $t_{\underline{m}}(\mathbb{T}) \subset t_{\underline{m}_A}(\mathbb{T})$ and $t_{\underline{m}}(\mathbb{T}) = \bigcap_{A\in[\mathbb{N}]^{\aleph_0}} t_{\underline{m}_A}(\mathbb{T})$.

Furthermore $\varphi(m_n x) = \varphi(\{m_n x\})$, and $\text{supp}_b(x) \subset \text{supp}(x)$.

Theorem 2.3. *Let $x \in [0, 1)$ with canonical representation (5) for a given $\underline{m}$ as in (1). Then $\varphi(x) \in t_{\underline{m}}(\mathbb{T})$ iff supp(x) is finite or if supp(x) is infinite and for all $A \in [\mathbb{N}]^{\aleph_0}$ the following holds:*

(a) If A is b-bounded then

(a_1) if $A \subset^* supp(x)$ then $A + 1 \subset^* supp(x)$, $A \subset^* supp_b(x)$ and $\lim_{n\in A} \frac{c_{n+1}+1}{b_{n+1}} = 1$ in $\mathbb{R}$.
Moreover if $A + 1$ is b-bounded, then $A + 1 \subset^* supp_b(x)$ as well;

(a_2) if $A \cap supp(x)$ is finite then $\lim_{n\in A} \frac{c_{n+1}}{b_{n+1}} = 0$ in $\mathbb{R}$.
Moreover if $A + 1$ is b-bounded, then $(A + 1) \cap supp(x)$ is finite as well.

(b) If A is b-divergent then $\lim_{n\in A} \varphi(\frac{c_n}{b_n}) = \lim_{n\in A} \varphi(\frac{c_n+1}{b_n}) = 0$ in $\mathbb{T}$.

Remark 2.4. Obviously, item (b) imposes the restriction only on $A \cap \text{supp}(x)$ (since $c_n = 0$ for all $n \notin \text{supp}(x)$). Hence, one can consider only subsets A of $\text{supp}(x)$ in item (b).

2.2 Preliminary steps

Let $x \in [0,\ 1)$ with canonical representation (5). For $n, t \in \mathbb{N}$ set

$$\mathfrak{S}_{n,t}(x) = \frac{c_n}{b_n} + \cdots + \frac{c_{n+t}}{b_n \cdots b_{n+t}}. \tag{6}$$

The motivation to introduce this "partial term" in the representation of $m_{n-1}x - [m_{n-1}]$ comes from the following formula:

Lemma 2.5. *[12, Lemma 3.1] For x represented as in (5), for every natural $n > 1$ and every non-negative integer t*

$$\{m_{n-1}x\} = \mathfrak{S}_{n,t}(x) + \frac{\{m_{n+t}x\}}{b_n \cdots b_{n+t}} \geq \mathfrak{S}_{n,t}(x). \tag{7}$$

For the sake of convenience sometimes we shall apply (7) with $t+1$ and split $\mathfrak{S}_{n,t+1}(x)$ in $\mathfrak{S}_{n,t}(x) + \frac{c_{n+t+1}}{b_n \cdots b_{n+t+1}}$ to get also

$$\{m_{n-1}x\} = \mathfrak{S}_{n,t+1}(x) + \frac{\{m_{n+t}x\}}{b_n \cdots b_{n+t+1}} = \mathfrak{S}_{n,t}(x) + \frac{c_{n+t+1}}{b_n \cdots b_{n+t+1}} + \frac{\{m_{n+t+1}x\}}{b_n \cdots b_{n+t+1}}. \tag{8}$$

Along with (7), this gives the following obvious, but useful estimate:

$$\mathfrak{S}_{n,t}(x) \leq \{m_{n-1}x\} \leq \mathfrak{S}_{n,t}(x) + \frac{c_{n+t+1}}{b_n \cdots b_{n+t+1}} + 2^{-(t+1)}. \tag{9}$$

Lemma 2.6. *If $A \in [\mathbb{N}]^{\aleph_0}$, $x \in [0,\ 1)$ and $\varphi(x) \in t_{\underline{m}_{A-1}}(\mathbb{T})$ then*

(i) if $A \subset^* supp(x)$ and b-bounded then $\lim_{n\in A} \{m_{n-1}x\} = 1$ in $\mathbb{R}$ and $A \subset^* supp_b(x)$

(ii) if $A \cap supp(x)$ is finite then $\lim_{n\in A} \{m_{n-1}x\} = 0$ in $\mathbb{R}$.

Proof. (i) Let $b := 1 + \max_{n\in A}\{b_n\}$. The hypothesis $A \subset^* \text{supp}(x)$ yields $c_n \geq 1$ for almost all $n \in A$. Since $\{m_{n-1}x\} \geq \frac{c_n}{b_n} \geq 1/b$ by Lemma 2.5 applied with $t = 0$, we conclude that

$$\{m_{n-1}x\} > \frac{1}{b} \text{ for almost all } n \in A. \tag{10}$$

Since $\varphi(x) \in t_{\underline{m}_{A-1}}(\mathbb{T})$, we conclude that $\lim_{n\in A} \{m_{n-1}x\} = 1$ in $\mathbb{R}$.

By Lemma 2.5, applied with $t = 0$, $\{m_{n-1}x\} = \frac{c_n}{b_n} + \frac{\{m_n x\}}{b_n}$ hence

$$1 - \frac{1}{b_n} < 1 - \frac{1}{b} < \{m_{n-1}x\} = \frac{c_n}{b_n} + \frac{\{m_n x\}}{b_n} < \frac{c_n + 1}{b_n}$$

for almost all $n \in A$ by (10). That is, $b_n - 1 < c_n + 1$, so $c_n = b_n - 1$ (as $c_n > b_n - 2$), for almost all $n \in A$. This proves $A \subset^* \text{supp}_b(x)$.

(ii) Again by Lemma 2.5 (with $t = 0$)) $\{m_{n-1}x\} = \frac{c_n}{b_n} + \frac{\{m_n x\}}{b_n}$. As $A \cap \text{supp}(x)$ is finite, $\{m_{n-1}x\} = 0 + \frac{\{m_n x\}}{b_n} \leq \frac{1}{2}$ for almost all $n \in A$. Since $\varphi(x) \in t_{\underline{m}_{A-1}}(\mathbb{T})$, we conclude that $\lim_{n\in A} \{m_{n-1}x\} = 0$. □

2.3 Proof of Theorem 2.3

Necessity. Suppose supp(x) is infinite and let $\varphi(x) \in t_{\underline{m}}(\mathbb{T})$ and $A \in [\mathbb{N}]^{\aleph_0}$.

(a) Suppose A is b-bounded and consider two cases.

(a_1) Let $A \subset^* \text{supp}(x)$. As $\varphi(x) \in t_{\underline{m}}(\mathbb{T})$ and A is b-bounded, Lemma 2.6 (1) entails $A \subset^* \text{supp}_b(x)$ and $1 = \lim_{n\in A} \{m_{n-1}x\}$. Hence, by Lemma 2.5, with $t = 0$,

$$1 = \lim_{n\in A} \left(\frac{c_n}{b_n} + \frac{\{m_n x\}}{b_n} \right) = \lim_{n\in A} \left(\frac{b_n - 1 + \{m_n x\}}{b_n} \right) = \lim_{n\in A} \left(1 - \frac{1 - \{m_n x\}}{b_n} \right)$$

This yields $\lim_{n\in A} \left(\frac{1 - \{m_n x\}}{b_n} \right) = 0$ and therefore

$$\lim_{n\in A} \{m_n x\} = 1, \tag{11}$$

as A is b-bounded. By the definition of canonical representation $c_{n+1} \leq b_{n+1} - 1$ for all $n \in \mathbb{N}$. By Lemma 2.5 (applied with $t = 0$) we have

$$\{m_n x\} = \frac{c_{n+1}}{b_{n+1}} + \frac{\{m_{n+1}x\}}{b_{n+1}} < \frac{c_{n+1} + 1}{b_{n+1}} \leq 1.$$

Hence (11) entails $1 = \lim_{n\in A} \{m_n x\} \le \lim_{n\in A} \frac{c_{n+1}+1}{b_{n+1}} \le 1$ i.e.

$$\lim_{n\in A} \frac{c_{n+1}+1}{b_{n+1}} = 1 \tag{12}$$

From (12) and the fact that $b_{n+1} \ge 2$ for each $n \in A$ we deduce that $c_{n+1}+1 > 1$ (i.e, $c_{n+1} \ne 0$) for almost all $n \in A$, i.e., $A+1 \subset^* \mathrm{supp}(x)$. By the first part of the proof, applied to $A+1$, we conclude that if $A+1$ is b-bounded this gives $A+1 \subset^* \mathrm{supp}_b(x)$.

(a$_2$) Let $A \cap \mathrm{supp}(x)$ by finite. By item (2) of 2.6, $\lim_{n\in A} \{m_{n-1}x\} = 0$ in $\mathbb{R}$, hence according to Lemma 2.5 (with $t = 1$)

$$0 = \lim_{n\in A} \{m_{n-1}x\} = \lim_{n\in A} \left(\frac{c_n}{b_n} + \frac{c_{n+1}}{b_n b_{n+1}} + \frac{\{m_{n+1}x\}}{b_n b_{n+1}} \right) = 0 + \lim_{n\in A} \left(\frac{c_{n+1}}{b_n b_{n+1}} + \frac{\{m_{n+1}x\}}{b_n b_{n+1}} \right) \tag{13}$$

Hence $\lim_{n\in A} \frac{\{m_{n+1}x\}}{b_n b_{n+1}} = \lim_{n\in A} \frac{c_{n+1}}{b_n b_{n+1}} = 0$ (for all $n \in \mathbb{N}$ we have $c_n, m_n \ge 0$ and $b_n > 0$). Finally, the b-boundedness of A yields $\lim_{n\in A} \frac{c_{n+1}}{b_{n+1}} = 0$.

If $A+1$ is b-bounded, the vanishing of the last limit implies that $(A+1) \cap \mathrm{supp}(x)$ is finite.

(b) Suppose A is b-divergent (i.e. $\lim_{n\in A} b_n = \infty$). By Lemma 2.5 (applied with $t = 0$) we get

$$0 = \lim_{n\in A} \varphi(\{m_{n-1}x\}) = \lim_{n\in A} \varphi\left(\frac{c_n}{b_n} + \frac{\{m_n x\}}{b_n} \right).$$

Along with $\{m_n x\} < 1$ and $\lim_{n\in A} b_n = \infty$,this yields $\lim_{n\in A} \varphi\left(\frac{c_n}{b_n}\right) = 0$.

Before starting the proof of the sufficiency let us reformulate the necessary conditions in a stronger iterated that will be frequently used in the sequel.

For any $A \in [\mathbb{N}]^{\aleph_0}$ and $t \in \mathbb{N}$ let $S_t(A) = \bigcup_{i=0}^t A + i$. Since our aim is to compute

$$\lim_{n\in A} \{m_{n-1}x\} = \lim_{n\in A} \mathfrak{S}_{n,t}(x) + \lim_{n\in A} \frac{\{m_{n+t}x\}}{b_n \cdots b_{n+t}} = \lim_{n\in A} \mathfrak{S}_{n,t}(x) + \lim_{n\in A} \frac{c_{n+t+1}}{b_n \cdots b_{n+t+1}} + \lim_{n\in A} \frac{\{m_{n+t+1}x\}}{b_n \cdots b_{n+t+1}}, \tag{14}$$

we use the second or the third part of (8) depending on whether there exists some t such that $S_t(A)$ is b-bounded, but $S_{t+1}(A)$ is not b-bounded. Note that in case such a t exists, one can assume without loss of generality that $A+t+1$ is actually b-divergent, by passing to an appropriate $A' \in [A]^{\aleph_0}$. We fix this in the following:

Claim 2.7. Suppose $x \in [0,1)$ with with canonical representation (5) such that (a), (b) hold. Let $A \in [\mathbb{N}]^{\aleph_0}$ is b-bounded. If $S_t(A)$ is b-bounded for some $t \in \mathbb{N}$, then

1. if $A \subset^* \mathrm{supp}(x)$ then $S_t(A) \subset^* \mathrm{supp}_b(x)$, $\lim_{n\in A+t+1} \frac{c_n+1}{b_n} = 1$ in $\mathbb{R}$ and there exists $n_t \in \mathbb{N}$ such that for all $n \ge n_t$

$$\mathfrak{S}_{n,t}(x) = 1 - \frac{1}{b_n \cdots b_{n+t}} \ge 1 - 2^{-(t+1)}. \tag{15}$$

   Moreover if $A+t+1$ is b-divergent, then

$$\lim_{n\in A+t+1} \frac{c_n}{b_n} = \lim_{n\in A} \frac{c_{n+t+1}}{b_{n+t+1}} = 1. \tag{16}$$

   and

$$\lim_{n\in A} \frac{\{m_{n+t+1}x\}}{b_n \cdots b_{n+t+1}} = 0. \tag{17}$$

2. if $A \cap \mathrm{supp}(x)$ is finite then $S_t(A) \cap \mathrm{supp}(x)$ is finite as well (so there exists $n_t \in \mathbb{N}$ such that $\mathfrak{S}_{n,t}(x) = 0$ for all $n \ge n_t$) and $\lim_{n\in A} \frac{c_{n+t+1}}{b_{n+t+1}} = 0$ in $\mathbb{R}$.

   Moreover if $A+t+1$ is b-divergent, then (17) holds true.

Sufficiency. If $\mathrm{supp}(x)$ is finite, let $n_0 := \max\{n \mid c_n \ne 0\}$. Then for all $n > n_0$ we get $m_n x \in \mathbb{Z}$ (due to 1), so $\varphi(x) \in t_{\underline{m}}(\mathbb{T})$.

Suppose now that $\mathrm{supp}(x)$ is infinite satisfying conditions (a) and (b); we have to prove that $\varphi(x) \in t_{\underline{m}}(\mathbb{T})$. For this purpose, according to Remark 2.2, we have to check that for all $A \in [\mathbb{N}]^{\aleph_0}$ there exists $A' \in [A]^{\aleph_0}$ such that $\lim_{n\in A'} \varphi(\{m_{n-1}x\}) = 0$. Hence, we can assume without loss of generality that either $A \subseteq \mathrm{supp}(x)$ or $A \cap \mathrm{supp}(x) = \emptyset$.

Suppose that A is b-bounded.

Consider first that case, when there exists an integer $t > 0$ such that $A + t + 1$ is not b-bounded, while for all $A + s$ is b-bounded for all $s \in \{n \in \mathbb{N} \mid n \leq t\}$, i.e., $S_t(A)$ is b-bounded. Without loss of generality we can assume that $A + t + 1$ is b-divergent (i.e., $\lim_{n\in A} b_{n+t+1} = \infty$). Then (17) holds, so (14) becomes

$$\lim_{n\in A} \{m_{n-1}x\} = \lim_{n\in A} \left(\mathfrak{S}_{n,t}(x) + \frac{c_{n+t+1}}{b_n \cdots b_{n+t+1}} \right) \tag{18}$$

In case $A' = A \cap \mathrm{supp}(x)$ is infinite, passing to A' we can assume without loss of generality that $A \subset^* \mathrm{supp}(x)$. Then Claim 2.7 and (16) imply

$$\lim_{n\in A} \{m_{n-1}x\} = \lim_{n\in A} \left(1 - \frac{1}{b_n \cdots b_{n+t}} + \frac{c_{n+t+1}}{b_n \cdots b_{n+t+1}} \right) = 1. \tag{19}$$

In case $A \cap \mathrm{supp}(x)$ is finite, Claim 2.7 applies again, so we get

$$\lim_{n\in A} \{m_{n-1}x\} = \lim_{n\in A} \frac{c_{n+t+1}}{b_n \cdots b_{n+t+1}} = 0. \tag{20}$$

Suppose that $S_t(A)$ is b-bounded for all $t \in \mathbb{N}$.

Let $\varepsilon > 0$. Pick a $t \in \mathbb{N}$ such that $2^{-(t+1)} < \varepsilon$. According to Claim 2.7 we can chose $n_t \in \mathbb{N}$ such that

(i) (15) holds for all $n > n_t$, in case $A \subseteq \mathrm{supp}(x)$; or

(ii) $\mathfrak{S}_{n,t} = 0$ and $\frac{c_{n+t+1}}{b_{n+t+1}} < \varepsilon$ hold for all $n > n_t$, in case $A \cap \mathrm{supp}(x) = \emptyset$.

In case (i), (15) from Claim 2.7 implies $\lim_{n\in A}\{m_{n-1}x\} \geq 1 - \varepsilon$ for all $n > n_t$, so $\lim_{n\in A}\{m_{n-1}x\} = 1$.

By (9) in case (ii) one has $\lim_{n\in A}\{m_{n-1}x\} \leq 2\varepsilon$ for all $n > n_t$, so $\lim_{n\in A}\{m_{n-1}x\} = 0$.

Finally, consider A not b-bounded; thence there exists $A' \in [A]^{\aleph_0}$ such that $\lim_{n\in A'} b_n = \infty$. By Lemma 2.5 (with $t = 0$),

$$\lim_{n\in A'} \varphi(\{m_{n-1}x\}) = \lim_{n\in A'} \varphi\left(\frac{c_n}{b_n} + \frac{\{m_n x\}}{b_n} \right). \tag{21}$$

Hence, by (b), the second limit in (9) is equal to $\lim_{n\in A'} \varphi\left(\frac{\{m_n x\}}{b_n}\right)$. Since $\{m_n x\} < 1$ and $\lim_{n\in A'} b_n = \infty$, we conclude that $\lim_{n\in A'} \varphi(\{m_{n-1}x\}) = 0$.

3 Some corollaries of Theorem 2.3

Due to its general character, Theorem 2.3 is somewhat heavy to apply directly. This is why we give now a series of corollaries where, under additional natural conditions, the description of the topologically $\underline{m}$-torsion elements of $\mathbb{T}$ becomes much more transparent.

3.1 Some restraints on supp(x)

The following simple claim will be needed in the proofs of the entire section.

Claim 3.1. If $A + 1 \subseteq^* A$ for an infinite subset A of $\mathbb{N}$, then A is co-finite subset of $\mathbb{N}$.

Proof. Fix a one-to-one increasing enumeration $A = \{n_k : k \in \mathbb{N}\}$. By our assumption there exists k_0 such that for all $k \geq k_0$ one has $n_k + 1 \in A$. Hence $n_{k_0} + 1 \in A$, so $n_{k_0} + 1 = n_{k_0+1} \in A$. Since $k_0 + 1 > k_0$, one has $n_{k_0+1} + 1 = (n_{k_0} + 1) + 1 \in A$. Analogously, $n_{k_0} + m \in A$ for all $m \in \mathbb{N}$, in other words A is co-finite. This proves the claim □

The next corollary gives a proof of Corollary 2.4 in [12] (its proof in [12] relies on the wrong Corollary 2.3 from that paper).

Corollary 3.2. [12, Corollary 2.4] *Let $x \in [0, 1)$. If supp(x) b-bounded, then the following are equivalent:*

(i) $\varphi(x) \in t_{\underline{m}}(\mathbb{T})$

(ii) $c_n = 0$ *for almost all* $n \in \mathbb{N}$

Proof. (i) → (ii) Suppose $\text{supp}(x)$ is infinite. By (a_1) of Theorem 2.3, setting $A = \text{supp}(x)$, we get $\text{supp}(x) + 1 \subseteq^* \text{supp}(x)$. Hence $\text{supp}(x)$ is co-finite by Claim 3.1. Then the whole set $\mathbb{N}$ is b-bounded. Therefore, (a_1) applied to $A = \mathbb{N}$ implies that $\text{supp}_b(x)$ is co-finite, a contradiction. □

Remark 3.3. A tempting counterpart of Corollary 3.2, stated in [12, Corollary 2.5], is: *if supp(x) is b-divergent, then* $\varphi(x) \in t_{\underline{m}}(\mathbb{T})$ *iff* $\lim_{n\in supp(x)} \frac{c_n}{b_n} = 0$ *in* $\mathbb{T}$. Unfortunately, it is false.

Now we give the correct counterpart of Corollary 3.2.

Corollary 3.4. *Suppose* $x \in [0, 1)$ *has b-divergent support. Then* $\varphi(x) \in t_{\underline{m}}(\mathbb{T})$ *iff the following two conditions are satisfied:*

(1) $\lim_{n\in supp(x)} \varphi\left(\frac{c_n}{b_n}\right) = 0$ *in* $\mathbb{T}$*; and*

(2) $\lim_{n\in I'} \frac{c_n}{b_n} = 0$ *in* $\mathbb{R}$ *for every infinite* $I' \subseteq supp(x)$ *such that* $I' - 1$ *is b-bounded.*

Proof. Let $I = \text{supp}(x)$. If $\varphi(x) \in t_{\underline{m}}(\mathbb{T})$, then (1) holds true by item (b) of Theorem 2.3 applied to $A = I$. Assume that $A = I' - 1$ is b-bounded for some infinite $I' \subseteq I$. Then $A \cap I$ is finite, as I is b-divergent. Then by (a_2) applied to A, $\lim_{n\in I'} \frac{c_n}{b_n} = \lim_{n\in A} \frac{c_{n+1}}{b_{n+1}} = 0$ in $\mathbb{R}$. This proves (2) and the necessity.

To establish the sufficiency, assume that (1) and (2) hold true. According to Theorem 2.3, to prove that $\varphi(x) \in t_{\underline{m}}(\mathbb{T})$ we have to check (a) and (b). Since (b) immediately follows from (1), we are left with (a). Let A be an infinite b-bounded set in $\mathbb{N}$. Then $A \cap I$ is finite, so we need to check only (a_2), i. e, $\lim_{n\in A} \frac{c_{n+1}}{b_{n+1}} = 0$ (since the final assertion of (a_2) follows from this equality, as mentioned in the final part of the proof of the necessity of (a_2)). Let $I' = (A + 1) \cap I$. If this set is infinite, then (2) applies and we are done. If I' is finite, we conclude that $c_n = 0$ for almost $n \in A + 1$ and hence $\lim_{n\in A+1} \frac{c_n}{b_n} = 0$. □

The following result was established in [15]:

Corollary 3.5. *Suppose* $x \in [0, 1)$ *has b-divergent support. Then* $\varphi(x) \in t_{\underline{m}}(\mathbb{T})$ *whenever* $\lim_{n\in supp(x)} \frac{c_n}{b_n} = 0$ *in* $\mathbb{R}$.

Follows immediately from Corollary 3.4 as the hypothesis implies both (1) and (2) from that corollary.
The next corollary, following obviously from the previous one, will be useful in the applications.

Corollary 3.6. *Suppose* $x \in [0, 1)$ *has b-divergent support. Then* $\varphi(x) \in t_{\underline{m}}(\mathbb{T})$ *whenever* (c_n) *is bounded.*

Now we obtain a result that generalizes Theorem 2.1:

Corollary 3.7. *Suppose that* $\mathbb{N}$ *is b-divergent. Then for* $x \in [0, 1)$ $\varphi(x) \in t_{\underline{m}}(\mathbb{T})$ *iff* $\lim_{n\in supp(x)} \varphi\left(\frac{c_n}{b_n}\right) = 0$ *in* $\mathbb{T}$.

Follows immediately from Corollary 3.4 as the hypothesis implies that no infinite b-bounded sets exist (so that (2) holds true vacuously).

Corollary 3.8. *For a sequence* $\underline{m}$ *as in (1) with* $m_n | m_{n+1}$ *for each* n *the following are equivalent:*

(a) $|t_{\underline{m}}(\mathbb{T})| = \mathfrak{c}$*;*

(b) $t_{\underline{m}}(\mathbb{T})$ *is uncountable;*

(c) $t_{\underline{m}}(\mathbb{T})$ *contains non-torsion elements;*

(d) $b_n = \frac{m_n}{m_{n-1}}$ *is not bounded.*

Proof. The implications (a) ⇒ (b) ⇒ (c) are trivial. The implication (c) ⇒ (d) follows from Corollary 3.2.

The implication (d) ⇒ (a) can be deduces directly from [4, Theorem 4.12], where one proves that if b_n is not bounded, then $|t_{\underline{m}}(\mathbb{T})| = \mathfrak{c}$ for a more general form of sequences which makes the proof of that theorem quite difficult. This is why, we shall deduce this implication from Corollary 3.6. Let $I \subseteq \mathbb{N}$ be a b-divergent set witnessing our hypothesis (d). For every infinite subset J of I let $x_J = \sum_{n\in J} \frac{1}{m_n}$. Then $x_J \neq x_{J'}$ whenever $J \neq J' \in [I]^\omega$, so the set $M = \{x_J : J \in [I]^\omega\}$ has size $\mathfrak{c}$. It remains to note that $M \subseteq t_{\underline{m}}(\mathbb{T})$ due to Corollary 3.6. □

Remark 3.9. It was claimed in [12] that the four equivalent conditions in the above corollary imply that $t_{\underline{m}}(\mathbb{T})$ is not divisible. The argument given there relies on the false [12, Corollary 2.5]. We have no proof at hand of this implication. Let us note that the same argument given in [12, Corollary 2.8] works in the case $b_n \to \infty$ for $n \in \mathbb{N}$.

3.2 Some restraints on the sequence (b_n)

Here we consider sequences (m_n) such that the sequence (b_n) splits into a b-bounded parts and a b-divergent part.

Definition 3.10. We say that the sequence (b_n) has the *splitting property* if exists a partition $\mathbb{N} = B \cup I$, such that

(a) B and I are either empty or infinite

(b) I is b-divergent, in case I is infinite;

(c) B is b-bounded.

We say that B and I witness the splitting property for (b_n). Note the B and I are uniquely defined up to a finite set (i.e., if $B' \cup I'$ is another partition witnessing the splitting property for (b_n), then $B' =^* B$ and $I' =^* I$).

Example 3.11. For every positive integer n write $n = 2^{b_n} n_1$, where n_1 is odd. Then the sequence (b_n) does not have the splitting property. (Indeed, it's easy to prove that a sequence (b_n) has the splitting property iff there exist n_0 such that for all $n \geq n_0$ the set $\{m : b_m = b_n\}$ is finite.)

Our next aim is to simplify Theorem 2.3 in the case when $b : n \mapsto b_n$ has the splitting property. The simplification consists in reducing the number of the infinite sets A in the text of that theorem, by using, for a fixed x, only *three* infinite sets, $B_S(x)$, $B_N(x)$ and $I_S(x)$, related to x, defined in the next

Notation. Let $x \in [0, 1)$ with canonical representation where $b : n \mapsto b_n$ has the splitting property. Let $B_S(x) = B \cap \mathrm{supp}(x)$, $B_N(x) = B \setminus B_S(x)$ and $I_S(x) = I \cap \mathrm{supp}(x)$. (to simplify the notation we are omitting the dependence from the sequence (m_n)).

According to Remark 2.4 the set $I \setminus I_S(x)$ will play no relevant role in the sequel. Note that the sets $B_S(x)$ and $B_N(x)$ are b-bounded, while $I_S(x)$ is b-divergent whenever it is infinite.

The next corollary is a characterization of topologically torsion elements of $\mathbb{T}$, in the case of $b_n = \frac{m_n}{m_{n-1}}$ has the splitting property.

Corollary 3.12. *Suppose that $b : n \mapsto b_n$ has the splitting property and let $x \in [0, 1)$ have canonical representation (5). Then $\varphi(x) \in t_{\underline{m}}(\mathbb{T})$ iff the following conditions hold:*

(i) $B_S(x) + 1 \subseteq^*$ supp(x), $B_S(x) \subseteq^*$ supp$_b(x)$ and if $B_S(x)$ is infinite, then $\lim_{n \in B_S(x)} \frac{c_{n+1}+1}{b_{n+1}} = 1$ in $\mathbb{R}$;

(ii) if $B_N(x)$ is infinite, then $\lim_{n \in B_N(x)} \frac{c_{n+1}}{b_{n+1}} = 0$ in $\mathbb{R}$.

(iii) if $I_S(x)$ is infinite, then $\lim_{n \in I_S(x)} \varphi\left(\frac{c_n}{b_n}\right) = 0$ in $\mathbb{T}$;

Proof. **Necessity.** Suppose $\varphi(x) \in t_{\underline{m}}(\mathbb{T})$, and hence (a) and (b) of Theorem 2.3 hold true. Let us check (i), (ii) and (iii).

(i) If $B_S(x)$ is finite there is nothing to prove. If $B_S(x)$ is infinite, then to get (i) it suffices to apply (a_1) of Theorem 2.3 to $A = B_S(x)$.

(ii) Assume $B_N(x)$ is infinite. By (a_2) of Theorem 2.3, applied to $A = B_N(x)$, one gets $\lim_{n \in A} \frac{c_{n+1}}{b_{n+1}} = 0$ in $\mathbb{R}$.

(iii) Suppose $I_S(x)$ infinite. By (b) of Theorem 2.3, applied to $A = I_S(x)$, one obtains $\lim_{n \in A} \varphi\left(\frac{c_n}{b_n}\right) = 0$ in $\mathbb{T}$.

Sufficiency. Suppose now (i), (ii) and (iii) hold, we have to check that (a) and (b) of theorem 2.3 hold too. Let $A \in [\mathbb{N}]^{\aleph_0}$.

(a) Suppose that A is b-bounded.

(a_1) If $A \subseteq^*$ supp(x), then $A \subseteq^* B_S(x)$ by the b-boundedness of A. Hence $B_S(x)$ is infinite. By (i), $B_S(x)+1 \subseteq^*$ supp(x), $B_S(x) \subseteq^*$ supp$_b(x)$ and $\lim_{n \in B_S(x)} \frac{c_{n+1}+1}{b_{n+1}} = 1$ in $\mathbb{R}$. Since $A \subseteq^* B_S(x)$, one has $A + 1 \subseteq^*$ supp(x), $A \subseteq^*$ supp$_b(x)$ and $\lim_{n \in A} \frac{c_{n+1}+1}{b_{n+1}} = 1$ in $\mathbb{R}$.

(a_2) If $A \cap$ supp(x) is finite, then $A \subseteq^* B_N(x)$ by the b-boundedness of A, hence $B_N(x)$ is infinite and $\lim_{n \in A} \frac{c_{n+1}}{b_{n+1}} = \lim_{n \in B_N(x)} \frac{c_{n+1}}{b_{n+1}}$. By (ii), the last limit is 0 in $\mathbb{R}$ therefore (a_2) of the theorem holds.

(b) Suppose that A is b-divergent. Then we can assume without loss of generality that $A \subseteq I(b)$. On the other hand, according to Remark 2.4, we can assume also that $A \subseteq$ supp(x). Hence $A \subseteq I_S(x)$. Since A is infinite, then also I_S is infinite. Hence $\lim_{n \in I_S} \varphi\left(\frac{c_n}{b_n}\right) = 0$ in $\mathbb{T}$ due to (iii). Consequently, $\lim_{n \in A} \varphi\left(\frac{c_n}{b_n}\right) = 0$. □

Corollary 3.13. *Suppose that $b : n \mapsto b_n$ has the splitting property. If $\varphi(x) \in t_{\underline{m}}(\mathbb{T})$ for some $x \in [0, 1)$ with infinite $B_S(x)$, then $I_S(x)$ is infinite.*

Proof. To prove the corollary let $A = B_S(x) + 1 \setminus B_S(x)$. By corollary 3.12

$$B_S(x) + 1 \subseteq^* \mathrm{supp}(x) \quad \text{and} \quad B_S(x) \subseteq^* \mathrm{supp}_b(x) \tag{22}$$

Then the first inclusion in (22) implies that $A \subseteq^* \mathrm{supp}(x) \setminus B_S(x) = I_S(x)$. So the corollary will be proved if we show that A is infinite.

Arguing for a contradiction assume that A is finite, i.e., $B_S(x) + 1 \subseteq^* B_S(x)$. By Claim 3.1, $B_S(x)$ is co-finite. Now the second inclusion in (22) yields that $\mathrm{supp}_b(x)$ is co-finite as well in contradiction with definition of canonical representation. □

As an application of the above corollary one can obtain a new proof of Corollary 3.2.

4 Counterexamples

The main difference between Theorem 2.3 and [12, Theorem 2.2] is item a$_2$) that was completely missing in [12]. Indeed, when A is bounded and almost contained in supp(x) (as in a$_1$), then the conclusion "$A \subset^* \mathrm{supp}_b(x)$ and $\lim_{n\in A} \frac{c_{n+1}+1}{b_{n+1}} = 1$ in $\mathbb{R}$" in the first part of item a$_1$) is obviously equivalent to the conclusion "$\lim_{n\in A} \frac{c_n+1}{b_n} = 1$ and $\lim_{n\in A} \frac{c_{n+1}+1}{b_{n+1}} = 1$ in $\mathbb{R}$" given in [12]. The next example explains the necessity to add a$_2$).

Example 4.1. As usual, set $(2k+1)!! = 1.3.\ldots(2k+1)$ and consider the sequences b_n, $m_n = b_1 \cdots b_n$ and c_n defined as follows:

$$b_n = \begin{cases} 2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd.} \end{cases}, \quad m_n = \begin{cases} 2^k(2k-1)!! & \text{if } n = 2k \text{ is even} \\ 2^k(2k+1)!! & \text{if } n = 2k+1 \text{ is odd} \end{cases} \quad \text{and} \quad c_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}. \tag{23}$$

Let $x = \sum_{n=1}^{\infty} \frac{c_n}{m_n}$.

With these data the hypothesis of [12, Theorem 2.2] is satisfied. Indeed, $\mathrm{supp}(x) = \{\text{odds numbers}\} \setminus \{1\}$ and so is infinite. If $A \subseteq \mathrm{supp}(x)$ is infinite, then A is b-divergent (i.e, $\lim_{n\in A} b_n = \infty$). From (23) we deduce

$$\lim_{n\in A} \frac{c_n}{b_n} = \lim_{n\in A} \frac{n-1}{n} = 0 \qquad \text{in } \mathbb{T}, \tag{24}$$

that is (b2) from [12, Theorem 2.2] holds true. Hence x verifies item (b) from [12, Theorem 2.2]. Yet $\varphi(x) \notin t_{\underline{m}}(\mathbb{T})$ as $m_{2k-1}\varphi(x) \to \frac{1}{2} \neq 0$ in $\mathbb{T}$.

Let us see now that (a$_2$) in Theorem 2.3 is not satisfied. Indeed, let A be the set of all even natural numbers. Then $A \cap \mathrm{supp}(x) = \emptyset$ is finite and A is b-bounded. Nevertheless,

$$\lim_{n\in A} \frac{c_{n+1}}{b_{n+1}} = \lim_{n\in A+1} \frac{c_n}{b_n} = \lim_{n\to\infty} \frac{n-1}{n} = 1 \qquad \text{in } \mathbb{R},$$

while this limit must equal 0 according to (a$_2$) of Theorem 2.3.

This example shows that also [12, corollary 2.3,2.5] are wrong. Indeed, as far as [12, corollary 2.3] is concerned, it suffices to note that with $I = \mathrm{supp}(x)$ the set of all odd naturals and $A = \emptyset$ the hypotheses of [12, corollary 2.3] are satisfied for x since $\lim_{n\in I} b_n = \infty$, (a) and (b) of [12, corollary 2.3] are vacuously satisfied, as $A = \emptyset$, while (c) is precisely (24). Nevertheless, $\varphi(x) \notin t_{\underline{m}}(\mathbb{T})$.

To see that [12, corollary 2.5] is wrong notice that with x as in the above example, supp(x) is b-divergent and (24) holds true, nevertheless, $\varphi(x) \notin t_{\underline{m}}(\mathbb{T})$.

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